NUMERICAL INVESTIGATION OF RETRIAL QUEUEING INVENTORY SYSTEM WITH A CONSTANT RETRIAL RATE, WORKING VACATION, FLUSH OUT, COLLISION AND IMPATIENT CUSTOMERS

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Abstract

The retrial queueing inventory system with working vacation, flush out, balking, breakdown, and repair, as well as a constant retrial rate and orbital client collision are all examined in this study. We made the assumption that customers arrive through a Markovian arrival process and that they would get phase-type services from the server. The inventory is replenished using a (s, S) and (s, Q) strategy, and it is expected that the replenishment time will follow an exponential distribution. If there are zero inventory items, no customers in the orbit, or both, the server will go into working vacation mode. When a customer retries an orbit while the server is serving arriving customers, the orbital customer may collide with the arriving customer during that retry, in which case both of them will be shifted back into orbit; otherwise, the orbital customer may avoid colliding with the arriving customer and may rejoin the orbit for another retry. The number of customers in the orbit and the inventory level may be found in the steady state. A cost analysis is produced along with the establishment of various important performance measures. Moreover, some numerical examples are provided to clarify our mathematical notion.

Keywords: Markovian arrival process, PH-distribution, working vacation, collision of orbital customers, flus out.

AMS Subject Classification (2010): 60K25, 68M30, 90B22.

1. Introduction

Retrial queues occur when initial consumers identify all servers and/or waiting space full. They may choose to try again after a random length of time or abandon the system permanently. RQ models have been thoroughly researched in a significan number of papers. Artalejo et al. [3] introduced the concept of retrial requests for inventory. They assumed that demand points are Poisson processes, whereas lead and retrial time points are exponential. They thought that the orbit's size is limitless. Manuel et al.[8] proposed a retrial inventory system that includes a service facility. They assumed clients come according to a Markovian arrival process (MAP), that service time for each client follows a phase-type distribution (PH), that lead time, lifetime of each item, and retrial times follow an exponential distribution.

Customers arrive at the single server retrial queueing-inventory system under consideration in this study using a Markovian Arrival Process, also known as the flexibl point process. The MAP tries to accomplish significan generalisation of the Poisson process while keeping it tractable. Many real-world applications do not require a renewal procedure before arriving. As a result, the most useful tool for simulating renewal and non-renewal appearance situations is the MAP. We can have realistic arrival patterns in this model because of the MAP, which also accounts for correlations and dependencies between arrivals. Furthermore, the continuous-time case is necessary, even though the MAP is define for both discrete and continuous periods. See Chakravarthy [5] and Neuts [10] for further details on the MAP and its properties.

The notion of server vacation was firs presented in the retry inventory system by Sivakumar[17]. For lead, inter-trial, inter-demand, and server vacation durations, he made the assumption that the distributions would be exponential. He also believed that these incidents are unrelated to one another. He instituted a programme of repeated vacations. A two-commodity substitutable retrial inventory system with a shared ordering strategy was examined by Sivakumar [15]. Sivakumar [16] examined a system of perishable inventory that had requests for retrials. The exponentially distributed lead periods for orders, the finit source of requests, the exponentially distributed life durations for stored objects, and the exponentially distributed inter-retrial intervals have all been assumed by the author. A two-commodity stochastic inventory technique with a complement item was proposed by Jeganathan et al. [11] in the context of a traditional retrial facility. When the primary item is out of supply, each new client will immediately enter an orbit of infinit capacity.

A $M/M/1$ retrial queue under (s, S) policy with a storage system was examined by Shajin and Krishnamoorthy [14]. The authors use the assumption that when the server is inactive, the external arrivals immediately enter an orbit and that the time between two successive retrials has an exponential distribution. Only the client at the head of the orbit is allowed to reach the server. In contrast to the traditional method of employing just one vendor, Chakravarthy and Hayat [6] established the idea of multiple vendors responsible for replacing inventories. This way, replenishment happens via two vendors. The authors used the MAM to analyse the model in steady-state under the assumptions of a two-vendor system, where the lead times are exponentially distributed with a parameter that depends on the vendor, the demands occur according to a MAP, and the service times are PH. There are also interesting numerical examples given, such as a comparison of the systems with one and two vendors.

A queueing inventory model in which a new customer comes and waits for service when the server is unavailable due to vacation was examined by Y Zhang et al. [19]. The model included the server's multiple vacations and dissatisfie clients. They were able to extract some significan performance metrics and fin the matrix geometric solution of the steady-state probability by using the truncated approximation approach. Using numerical analysis, the impact of the probability and impatience rate on a few performance metrics was examined. Using the genetic algorithm, the authors calculated the best possible policy and cost and arrived at the ideal service rate. Ayyappan et al. [4] studied the notions of working breakdown, collision, vacation, and reneging in a non-preemptive priority retrial queueing system with immediate feedback. They applied the supplementary variable technique to their model and also provided particular cases.

Service interruptions were originally implemented in an inventory model by Krishnamoorthy et al. [7]. They also believed that orders are processed instantly and that there is no limit to the amount of disruptions that can happen during a single service. Ushakumari [18] examined a (s, S) inventory system with recurrent demands for unfulfille requests from the orbit and a random lead time. In their paper [1], Amirthakodi and Sivakumar spoke about retrial inventory queueing with a single server and customer feedback, where the orbit size is finite.Th retrial queueing model with exponential service time, Poisson arrival, and delayed feedback was examined by Melikov et al. [9]. They used both (s, S) and (s, Q) replenishment policies for their study. In their analysis of an $M/M/1/N$ queuing system with reverse balking, Kumar et al. [13] incorporate the idea of reverse reneging. Customers' input is used by Kumar and Som [**?**] in an $M/M/1/N$ queuing system with reverse balking, reverse reneging, and retention of reneged customers. They calculate the system size stationary probability.

2. Model Description

- We examine a single-server retrial queueing inventory model in which customers arrive at the system as represented by MAP, with D_0 and D_1 matrices as its dimension m. The service times, denoted as (γ, U) of order n, are assumed to follow the PH-distribution with $U^0 + Ue = 0$.
- If the server is available, he serves the customer right away upon their arrival. If not, the customer must enter the orbit of infinit . Every customer retries from the orbit at a constant rate, despite the size of the orbit. The inter-retrial times follow an exponential distribution with parameter δ.
- If the orbit is empty, the inventory is zero, or both, then the server goes on vacation after serving the customer. Additionally , the vacation periods are expected to follow a η-parameter exponential distribution. In the event that a customer arrives during vacation time, the server will start charging the customer less for services than usual. Additionally , it is expected that the service times throughout the vacation period follow the PH distribution, denoted as $(\gamma, \theta U)$, with $0 < \theta < 1$. If the server examines the customer who is waiting in the system after completing this vacation, he will begin a normal busy period. Otherwise, he is dormant.
- The incoming customer may enter the orbit for a retry with probability q_1 or balk the system with probability p_1 during the service delivery, repair, and no inventory items, ensuring that $p_1 + q_1 = 1$.
- When a customer retries an orbit while the server is servicing incoming customers, there is a chance that the orbital customer and the incoming customer will collide and be shifted to the orbit with a probability of q_2; if not, the orbital customer may not collide and will rejoin the orbit for a subsequent retry with a probability of p_2, such that $p_2 + q_2 = 1$.
- During regular busy periods, the server may get breakdown. As a result, the customer getting service at the moment must enter the orbit of limitless capacity. The server goes into idle mode when the repair operation is completed. The breakdown times are exponentially distributed with parameter ψ, whereas the repair times are PH-distributed with rate (α, T).
- All the customers in the orbit are flushe out periodically and the flus out times follow exponential distribution with parameter σ. The schematic picture of this model is provided in Figure 1.
- $\otimes$ - Kronecker product of two matrices of different dimensions. $\oplus$ - Kronecker sum of two matrices of different dimensions. e - Column vector has an suitable size with each of its entries as 1. **0** - It denotes zero matrices in the suitable order.

3. Analysis

In the following section, we establish the queueing-inventory system's transition rate matrix. Assume that $N(t), J(t), I(t), R(t), S(t), A(t)$ describe the total customers in the orbit, status of server, stock level, repair phases, service phases, arrival phases, respectively.

$$J(t) = \begin{cases} 0, & \text{server is idle in normal service mode,} \\ 1, & \text{server is busy in normal service mode,} \\ 2, & \text{server is idle in WV mode,} \\ 3, & \text{server is busy in WV mode,} \\ 4, & \text{server is repair mode.} \end{cases}$$

Consider $X(t) = \{N(t), J(t), I(t), R(t), S(t), A(t)\}$ is a $CTMC$ with state space

$$\Phi = \phi(0) \bigcup_{i=1}^{\infty} \phi(i). \tag{1}$$

where

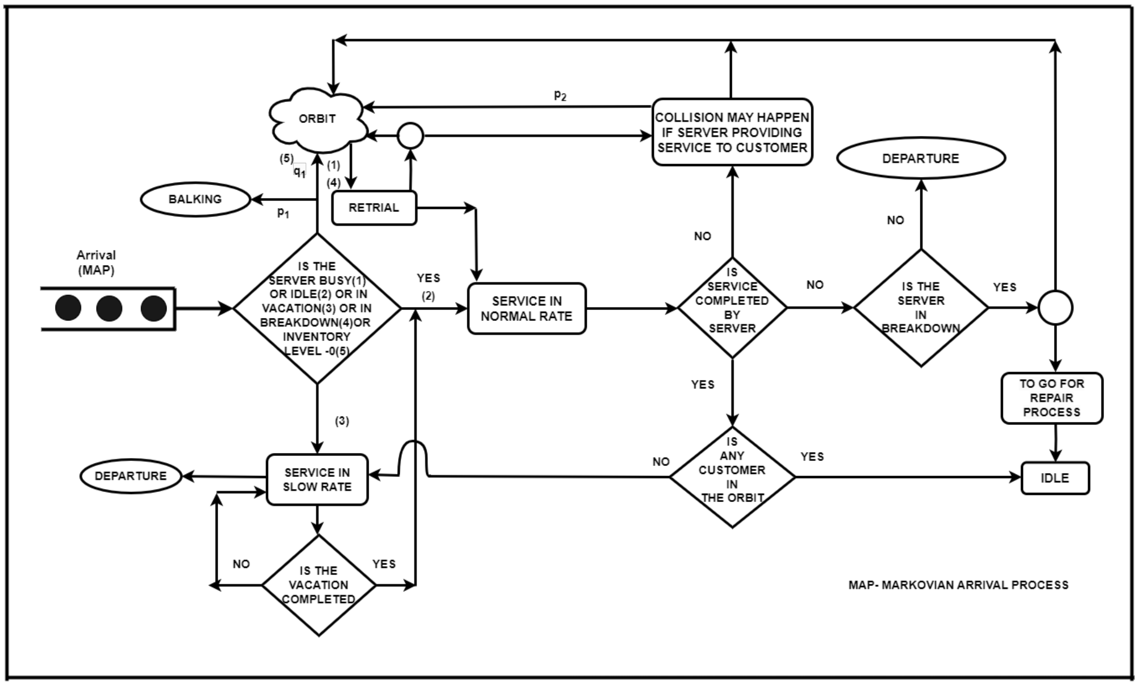

Figure 1: *Schematic representation*

$$
\begin{aligned}
\phi(0) = & \{(0,0,u_1,u_4): \ 0 \le u_1 \le S, \ 1 \le u_4 \le m\} \\
& \cup \{(0,1,u_1,u_3,u_4): \ 1 \le u_1 \le S, \ 1 \le u_3 \le n, \ 1 \le u_4 \le m\} \\
& \cup \{(0,2,u_1,u_4): \ 0 \le u_1 \le S, \ 1 \le u_4 \le m\} \\
& \cup \{(0,3,u_1,u_3,u_4): \ 1 \le u_1 \le S, \ 1 \le u_3 \le n, \ 1 \le u_4 \le m\} \\
& \cup \{(0,4,u_1,u_2,u_4): \ 1 \le u_1 \le S, \ 1 \le u_2 \le l, \ 1 \le u_4 \le m\}
\end{aligned}
$$

and for $i \ge 1$,

$$
\begin{aligned}
\phi(i) = & \{(i,0,u_1,u_4): \ 0 \le u_1 \le S, \ 1 \le u_4 \le m\} \\
& \cup \{(i,1,u_1,u_3,u_4): \ 1 \le u_1 \le S, \ 1 \le u_3 \le n, \ 1 \le u_4 \le m\} \\
& \cup \{(i,2,u_1,u_4): \ 0 \le u_1 \le S, \ 1 \le u_4 \le m\} \\
& \cup \{(i,3,u_1,u_3,u_4): \ 1 \le u_1 \le S, \ 1 \le u_3 \le n, \ 1 \le u_4 \le m\} \\
& \cup \{(i,4,u_1,u_2,u_4): \ 1 \le u_1 \le S, \ 1 \le u_2 \le l, \ 1 \le u_4 \le m\}
\end{aligned}
$$

3.1. Construction of the QBD process for Model 1

The generator matrix of the Markov chain under (s,S) policy is given by:

$$
\mathbb{Q} = \begin{bmatrix}
A_{00} & A_{01} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \dots \\
A_{10} & F_1 & F_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \dots \\
A & F_2 & F_1 & F_0 & \mathbf{0} & \mathbf{0} & \dots & \dots \\
A & \mathbf{0} & F_2 & F_1 & F_0 & \mathbf{0} & \dots & \dots \\
A & \mathbf{0} & \mathbf{0} & F_2 & F_1 & F_0 & \dots & \dots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \dots & \dots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \dots & \dots
\end{bmatrix}
$$

The entries in the block matrices of $\mathbb{Q}$ are define as follows,

$$A_{00} = \begin{bmatrix} A_{00}^{11} & A_{00}^{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_{00}^{22} & A_{00}^{23} & \mathbf{0} & \mathbf{0} \\ A_{00}^{31} & \mathbf{0} & A_{00}^{33} & A_{00}^{34} & \mathbf{0} \\ \mathbf{0} & A_{00}^{42} & A_{00}^{43} & A_{00}^{44} & \mathbf{0} \\ A_{00}^{51} & \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{00}^{55} \end{bmatrix},$$

where

$$A_{00}^{11} = \begin{bmatrix} C_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_2 \\ \mathbf{0} & C_3 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_2 \\ \mathbf{0} & \mathbf{0} & C_3 & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_3 & \mathbf{0} & \dots & \mathbf{0} & C_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_4 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_4 \end{bmatrix},$$

$$A_{00}^{22} = \begin{bmatrix} C_5 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_6 \\ \mathbf{0} & C_5 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_6 \\ \mathbf{0} & \mathbf{0} & C_5 & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_6 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_5 & \mathbf{0} & \dots & \mathbf{0} & C_6 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_7 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_7 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_7 \end{bmatrix},$$

where $C_1 = (D_0 + p_1 D_1) - \beta I_m$, $C_2 = \beta I_m$, $C_3 = D_0 - \beta I_m$, $C_4 = D_0$,
$C_5 = U \oplus (D_0 + p_1 D_1) - (\psi + \beta) I_{nm}$, $C_6 = \beta I_{nm}$, $C_7 = U \oplus (D_0 + p_1 D_1) - \psi I_{nm}$.
$A_{00}^{23} = I_S \otimes U^0 \otimes I_m$, $A_{00}^{31} = I_{S+1} \otimes \eta I_m$,

$$A_{00}^{12} = \begin{bmatrix} \mathbf{0} \\ I_S \otimes \gamma \otimes D_1 \end{bmatrix},$$

$$A_{00}^{33} = \begin{bmatrix} C_8 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_9 \\ \mathbf{0} & C_{10} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_9 \\ \mathbf{0} & \mathbf{0} & C_{10} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_9 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{10} & \mathbf{0} & \dots & \mathbf{0} & C_9 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{11} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{11} \end{bmatrix},$$

where $C_8 = (D_0 + p_1 D_1) - (\eta + \beta) I_m$, $C_9 = \beta I_m$, $C_{10} = D_0 - (\eta + \beta) I_m$, $C_{11} = D_0 - \eta I_m$.
$A_{00}^{42} = I_{S+1} \otimes \eta I_{nm}$, $A_{00}^{43} = I_S \otimes \theta U^0 \otimes I_m$,

$$A_{00}^{34} = \begin{bmatrix} \mathbf{0} \\ I_S \otimes \gamma \otimes D_1 \end{bmatrix}, A_{00}^{44} = \begin{bmatrix} C_{12} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{13} \\ \mathbf{0} & C_{12} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{13} \\ \mathbf{0} & \mathbf{0} & C_{12} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{13} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{12} & \mathbf{0} & \dots & \mathbf{0} & C_{13} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{14} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{14} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{14} \end{bmatrix},$$

where $C_{12} = \theta U \oplus (D_0 + p_1 D_1) - (\eta + \beta) I_{nm}$, $C_{13} = \beta I_{nm}$, $C_{14} = \theta U \oplus (D_0 + p_1 D_1) - \eta I_{nm}$.

$$A_{00}^{51} = \begin{bmatrix} \mathbf{0} & I_S \otimes T^0 \otimes I_m \end{bmatrix}, A_{00}^{55} = \begin{bmatrix} C_{15} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{16} \\ \mathbf{0} & C_{15} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{16} \\ \mathbf{0} & \mathbf{0} & C_{15} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{16} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{15} & \mathbf{0} & \dots & \mathbf{0} & C_{16} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{17} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{17} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{17} \end{bmatrix},$$

where $C_{15} = T \oplus (D_0 + p_1 D_1) - \beta I_{lm}$, $C_{16} = \beta I_{lm}$, $C_{17} = T \oplus (D_0 + p_1 D_1)$.

$$A_{01} = \begin{bmatrix} A_{01}^{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_{01}^{22} & \mathbf{0} & \mathbf{0} & A_{01}^{25} \\ \mathbf{0} & \mathbf{0} & A_{01}^{33} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{01}^{44} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{01}^{55} \end{bmatrix},$$

$A_{01}^{11} = \begin{bmatrix} q_1 D_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $A_{01}^{22} = I_S \otimes I_n \otimes q_1 D_1$, $A_{01}^{25} = I_S \otimes e_n \alpha \otimes \psi I_m$, $A_{01}^{33} = \begin{bmatrix} q_1 D_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$,
$A_{01}^{44} = I_S \otimes I_n \otimes q_1 D_1$, $A_{01}^{55} = I_S \otimes I_l \otimes q_1 D_1$,

$$A_{10} = \begin{bmatrix} A_{10}^{11} & A_{10}^{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A_{10}^{21} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_{10}^{33} & A_{10}^{34} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_{10}^{43} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{10}^{55} \end{bmatrix},$$

where
$A_{10}^{11} = I_{S+1} \otimes \sigma I_m$, $A_{10}^{12} = \begin{bmatrix} \mathbf{0} \\ I_S \otimes \delta\gamma \otimes I_m \end{bmatrix}$, $A_{10}^{21} = \begin{bmatrix} \mathbf{0} & I_S \otimes e_n \otimes \sigma I_m \end{bmatrix}$, $A_{10}^{33} = I_{S+1} \otimes \sigma I_m$,
$A_{10}^{34} = \begin{bmatrix} \mathbf{0} \\ I_S \otimes \delta\gamma \otimes I_m \end{bmatrix}$, $A_{10}^{43} = \begin{bmatrix} \mathbf{0} & I_S \otimes e_n \otimes \sigma I_m \end{bmatrix}$, $A_{10}^{55} = I_S \otimes \sigma I_{lm}$.

$$F_1 = \begin{bmatrix} F_1^{11} & F_1^{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ F_1^{21} & F_1^{22} & F_1^{23} & \mathbf{0} & \mathbf{0} \\ F_1^{31} & \mathbf{0} & F_1^{33} & F_1^{34} & \mathbf{0} \\ \mathbf{0} & F_1^{42} & F_1^{43} & F_1^{44} & \mathbf{0} \\ F_1^{51} & \mathbf{0} & \mathbf{0} & \mathbf{0} & F_1^{55} \end{bmatrix},$$

wher e $$F_1^{11} = \begin{bmatrix} C_{18} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{19} \\ \mathbf{0} & C_{20} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{19} \\ \mathbf{0} & \mathbf{0} & C_{20} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{19} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{20} & \mathbf{0} & \dots & \mathbf{0} & C_{19} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{21} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{21} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{21} \end{bmatrix},$$

wher e $C_{18} = (D_0 + p_1 D_1) - (\sigma + \beta) I_m$, $C_{19} = \beta I_m$, $C_{20} = D_0 - (\delta + \sigma + \beta) I_m$,
$C_{21} = D_0 - (\delta + \sigma) I_m$. $F_1^{12} = \begin{bmatrix} \mathbf{0} \\ I_S \otimes \gamma \otimes D_1 \end{bmatrix}$, $F_1^{21} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{S-1} \otimes U^0 \otimes I_m & \mathbf{0} \end{bmatrix}$,

$$F_1^{23} = \begin{bmatrix} U^0 \otimes I_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad F_1^{22} = \begin{bmatrix} C_{22} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{23} \\ \mathbf{0} & C_{22} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{23} \\ \mathbf{0} & \mathbf{0} & C_{22} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{23} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{22} & \mathbf{0} & \dots & \mathbf{0} & C_{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{24} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{24} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{24} \end{bmatrix},$$

wher e $C_{22} = U \oplus (D_0 + p_1 D_1) + [(q_2\delta - \delta) - (\psi + \sigma + \beta)] I_{nm}$, $C_{23} = \beta I_{nm}$,
$C_{24} = U \oplus (D_0 + p_1 D_1) + [(q_2\delta - \delta) - (\psi + \sigma)] I_{nm}$, $F_1^{31} = I_{S+1} \otimes \eta I_m$,

$$F_1^{34} = \begin{bmatrix} \mathbf{0} \\ I_S \otimes \gamma \otimes D_1 \end{bmatrix}, \quad F_1^{33} = \begin{bmatrix} C_{25} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{26} \\ \mathbf{0} & C_{27} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{26} \\ \mathbf{0} & \mathbf{0} & C_{27} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{26} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{27} & \mathbf{0} & \dots & \mathbf{0} & C_{26} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{28} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{28} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{28} \end{bmatrix},$$

wher e $C_{25} = (D_0 + p_1 D_1) - (\sigma + \eta + \beta) I_m$, $C_{26} = \beta I_m$, $C_{27} = D_0 - (\sigma + \delta + \eta + \beta) I_m$,
$C_{28} = D_0 - (\sigma + \delta + \eta) I_m$, $F_1^{42} = I_{S+1} \otimes \eta I_{nm}$, $F_1^{43} = \begin{bmatrix} I_S \otimes \theta U^0 \otimes I_m & \mathbf{0} \end{bmatrix}$,

$$F_1^{44} = \begin{bmatrix} C_{29} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{30} \\ \mathbf{0} & C_{29} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{30} \\ \mathbf{0} & \mathbf{0} & C_{29} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{30} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{29} & \mathbf{0} & \dots & \mathbf{0} & C_{30} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{31} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{31} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{31} \end{bmatrix},$$

wher e $C_{29} = \theta U \oplus (D_0 + p_1 D_1) + [(q_2\delta - \delta) - (\sigma + \eta + \beta)] I_{nm}$, $C_{30} = \beta I_{nm}$,
$C_{31} = \theta U \oplus (D_0 + p_1 D_1) + [(q_2\delta - \delta) - (\sigma + \eta)] I_{nm}$.

$$F_1^{51} = \begin{bmatrix} \mathbf{0} & I_S \otimes T^0 \otimes I_m \end{bmatrix}, F_1^{55} = \begin{bmatrix} C_{32} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{33} \\ \mathbf{0} & C_{32} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{33} \\ \mathbf{0} & \mathbf{0} & C_{32} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{33} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{32} & \mathbf{0} & \dots & \mathbf{0} & C_{33} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{34} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{34} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{34} \end{bmatrix},$$

where $C_{32} = T \oplus (D_0 + p_1 D_1) - (\sigma + \beta) I_{lm}$, $C_{33} = \beta I_{lm}$, $C_{34} = T \oplus (D_0 + p_1 D_1) - \sigma I_{lm}$.

$$F_0 = \begin{bmatrix} F_0^{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ F_0^{21} & F_0^{22} & \mathbf{0} & \mathbf{0} & F_0^{25} \\ \mathbf{0} & \mathbf{0} & F_0^{33} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & F_0^{43} & F_0^{44} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & F_0^{55} \end{bmatrix},$$

$F_0^{11} = \begin{bmatrix} q_1 D_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $F_0^{21} = \begin{bmatrix} \mathbf{0} & I_S \otimes e_n \otimes p_2 \delta I_m \end{bmatrix}$, $F_0^{22} = I_S \otimes I_n \otimes q_1 D_1$, $F_0^{25} = I_S \otimes e_n \alpha \otimes \psi I_m$,
$F_0^{33} = \begin{bmatrix} q_1 D_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $F_0^{43} = \begin{bmatrix} \mathbf{0} & I_S \otimes e_n \otimes p_2 \delta I_m \end{bmatrix}$, $F_0^{44} = I_S \otimes I_n \otimes q_1 D_1$, $F_0^{55} = I_S \otimes I_l \otimes q_1 D_1$.

$$F_2 = \begin{bmatrix} \mathbf{0} & F_2^{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & F_2^{34} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

where $F_2^{12} = \begin{bmatrix} \mathbf{0} \\ I_S \otimes \delta\gamma \otimes I_m \end{bmatrix}$, $F_2^{34} = \begin{bmatrix} \mathbf{0} \\ I_S \otimes \delta\gamma \otimes I_m \end{bmatrix}$,

$$A = \begin{bmatrix} A^{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A^{21} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A^{33} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A^{43} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & A^{55} \end{bmatrix},$$

where $A^{11} = I_{S+1} \otimes \sigma I_m$, $A^{21} = \begin{bmatrix} \mathbf{0} & I_S \otimes e_n \otimes \sigma I_m \end{bmatrix}$, $A^{33} = I_{S+1} \otimes \sigma I_m$,
$A^{43} = \begin{bmatrix} \mathbf{0} & I_S \otimes e_n \otimes \sigma I_m \end{bmatrix}$, $A_{10}^{53} = \begin{bmatrix} \mathbf{0} & I_S \otimes e_l \otimes \sigma I_m \end{bmatrix}$. $A^{55} = I_S \otimes \sigma I_{lm}$,

Stability condition for Model I

To discuss the stability condition, we firs consider the generator matrix $F = F_0 + F_1 + F_2$. If $\chi = (\chi_0, \chi_1, \chi_2, \chi_3, \chi_4)$=$(\chi_{00}, \chi_{01}, \dots, \chi_{0s}, \chi_{0s+1}, \dots, \chi_{0S}, \chi_{11}, \chi_{12}, \dots, \chi_{1s}, \chi_{1s+1}, \dots, \chi_{1S}, \chi_{20},$ $\chi_{21}, \dots, \chi_{2s}, \chi_{2s+1}, \dots, \chi_{2S}, \chi_{31}, \chi_{32}, \dots, \chi_{3s}, \chi_{3s+1}, \dots, \chi_{3S}, \chi_{41}, \chi_{42}, \dots, \chi_{4s}, \chi_{4s+1}, \dots, \dots, \chi_{4S})$.

The vector χ represents the invariant vector of matrix F. Consequently, we have the relations $\chi F = 0$ and $\chi e = 1$. For the Markov process with a QBD structure to exhibit stability, our model must satisfy the condition $\chi F_0 e < \chi F_2 e$. This condition is both necessary and sufficien for the stability of the queueing model under study and reduces to the inequality $\lambda < \mu$.

3.2. QBD process for Model II

In accordance with the assumptions outlined in the "Model Description" section, we will now examine Model II, while solely modifying the ordering policy from (s, S) to (s, Q). The generator matrix of the process for the (s, Q) policy takes on the following form:

$$\tilde{\mathbb{Q}} = \begin{bmatrix} \tilde{A}_{00} & A_{01} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \dots \\ A_{10} & \tilde{F}_1 & F_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \dots \\ A & F_2 & \tilde{F}_1 & F_0 & \mathbf{0} & \mathbf{0} & \dots & \dots \\ A & \mathbf{0} & F_2 & \tilde{F}_1 & F_0 & \mathbf{0} & \dots & \dots \\ A & \mathbf{0} & \mathbf{0} & F_2 & \tilde{F}_1 & F_0 & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \dots & \dots \end{bmatrix}$$

The entries in the block matrices of $\tilde{\mathbb{Q}}$ are define as follows,

$$\tilde{A}_{00} = \begin{bmatrix} \tilde{A}_{00}^{11} & \tilde{A}_{00}^{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{A}_{00}^{22} & \tilde{A}_{00}^{23} & \mathbf{0} & \mathbf{0} \\ \tilde{A}_{00}^{31} & \mathbf{0} & \tilde{A}_{00}^{33} & \tilde{A}_{00}^{34} & \mathbf{0} \\ \mathbf{0} & \tilde{A}_{00}^{42} & \tilde{A}_{00}^{43} & \tilde{A}_{00}^{44} & \mathbf{0} \\ \tilde{A}_{00}^{51} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{A}_{00}^{55} \end{bmatrix},$$

$$\tilde{A}_{00}^{11} = \begin{bmatrix} C_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_2 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_3 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_2 & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_3 & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_3 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_4 & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_4 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_4 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_4 \end{bmatrix},$$

$\tilde{A}_{00}^{12} = \begin{bmatrix} \mathbf{0} \\ I_S \otimes \gamma \otimes D_1 \end{bmatrix}$, $\tilde{A}_{00}^{23} = I_S \otimes U^0 \otimes I_m$, $\tilde{A}_{00}^{31} = I_{S+1} \otimes \eta I_m$,

$$\tilde{A}_{00}^{22} = \begin{bmatrix} C_5 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_6 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_5 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_6 & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_5 & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_5 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_6 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_7 & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_7 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_7 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_7 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_7 \end{bmatrix},$$

$$\tilde{A}_{00}^{33}=\begin{bmatrix}C_8&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&C_9&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&C_{10}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&C_9&\dots&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&C_{10}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\dots&C_{10}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&C_9\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&C_{11}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&C_{11}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&C_{11}&\dots&\mathbf{0}&\mathbf{0}\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&C_{11}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&C_{11}\end{bmatrix},$$

$$\tilde{A}_{00}^{34}=\begin{bmatrix}\mathbf{0}\\ I_S\otimes\gamma\otimes D_1\end{bmatrix},\ \tilde{A}_{00}^{42}=I_{S+1}\otimes\eta I_{nm},\ \tilde{A}_{00}^{43}=I_S\otimes\theta U^0\otimes I_m,\ \tilde{A}_{00}^{51}=\begin{bmatrix}\mathbf{0} & I_S\otimes T^0\otimes I_m\end{bmatrix},$$

$$\tilde{A}_{00}^{44}=\begin{bmatrix}C_{12}&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&C_{13}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&C_{12}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&C_{13}&\dots&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&C_{12}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\dots&C_{12}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&C_{13}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&C_{14}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&C_{14}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&C_{14}&\dots&\mathbf{0}&\mathbf{0}\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&C_{14}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&C_{14}\end{bmatrix},$$

$$\tilde{A}_{00}^{55}=\begin{bmatrix}C_{15}&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&C_{16}\\ \mathbf{0}&C_{15}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&C_{16}\\ \mathbf{0}&\mathbf{0}&C_{15}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&C_{16}\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\vdots\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\dots&C_{15}&\mathbf{0}&\dots&\mathbf{0}&C_{16}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&C_{17}&\dots&\mathbf{0}&\mathbf{0}\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\vdots\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&C_{17}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&\mathbf{0}&\dots&\mathbf{0}&C_{17}\end{bmatrix},$$

$$\tilde{F}_1=\begin{bmatrix}\tilde{F}_1^{11}&\tilde{F}_1^{12}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \tilde{F}_1^{21}&\tilde{F}_1^{22}&\tilde{F}_1^{23}&\mathbf{0}&\mathbf{0}\\ \tilde{F}_1^{31}&\mathbf{0}&\tilde{F}_1^{33}&\tilde{F}_1^{34}&\mathbf{0}\\ \mathbf{0}&\tilde{F}_1^{42}&\tilde{F}_1^{43}&\tilde{F}_1^{44}&\mathbf{0}\\ \tilde{F}_1^{51}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\tilde{F}_1^{55}\end{bmatrix},$$

wher e

$$\tilde{F}_1^{11} = \begin{bmatrix} C_{18} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{19} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{20} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{19} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{20} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{20} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{19} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{21} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{21} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{21} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{21} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{21} \end{bmatrix},$$

$\tilde{F}_1^{12} = \begin{bmatrix} \mathbf{0} \\ I_S \otimes \gamma \otimes D_1 \end{bmatrix}$, $\tilde{F}_1^{21} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{S-1} \otimes U^0 \otimes I_m & \mathbf{0} \end{bmatrix}$, $\tilde{F}_1^{23} = \begin{bmatrix} U^0 \otimes I_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$,
$\tilde{F}_1^{31} = I_{S+1} \otimes \eta I_m$, $\tilde{F}_1^{34} = \begin{bmatrix} \mathbf{0} \\ I_S \otimes \gamma \otimes D_1 \end{bmatrix}$,

$$\tilde{F}_1^{22} = \begin{bmatrix} C_{22} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{23} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{22} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{23} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{22} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{22} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{24} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{24} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{24} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{24} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{24} \end{bmatrix},$$

$$\tilde{F}_1^{33} = \begin{bmatrix} C_{25} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{26} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{27} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{26} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{27} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{27} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{26} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{28} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{28} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{28} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{28} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{28} \end{bmatrix},$$

$\tilde{F}_1^{42} = I_{S+1} \otimes \eta I_{nm}$, $\tilde{F}_1^{43} = \begin{bmatrix} I_S \otimes \theta U^0 \otimes I_m & \mathbf{0} \end{bmatrix}$,

$$\tilde{F}_1^{44} = \begin{bmatrix} C_{29} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{30} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{29} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{30} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{29} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{29} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{30} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{31} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{31} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{31} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{31} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{31} \end{bmatrix},$$

$\tilde{F}_1^{51} = \begin{bmatrix} \mathbf{0} & I_S \otimes T^0 \otimes I_m \end{bmatrix}$,

$$\tilde{F}_1^{55} = \begin{bmatrix} C_{32} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{33} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{32} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{33} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{32} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{32} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{33} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{34} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{34} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{34} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{34} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{34} \end{bmatrix},$$

Stability condition for Model II

To discuss the stability condition, we firs consider the generator matrix $F = F_0 + \tilde{F}_1 + F_2$. If $\chi = (\chi_0, \chi_1, \chi_2, \chi_3, \chi_4)$=$(\chi_{00}, \chi_{01}, \dots, \chi_{0s}, \chi_{0s+1}, \dots, \chi_{0Q}, \dots, \chi_{0S}, \chi_{11}, \chi_{12}, \dots, \chi_{1s}, \chi_{1s+1}, \dots, \chi_{1Q} \dots, \chi_{1S}, \chi_{20}, \chi_{21}, \dots, \chi_{2s}, \chi_{2s+1}, \dots, \chi_{2Q}, \dots, \chi_{2S}, \chi_{31}, \chi_{32}, \dots, \chi_{3s}, \chi_{3s+1}, \dots, \chi_{3Q}, \dots, \chi_{3S}, \chi_{41}, \chi_{42}, \dots, \chi_{4s}, \chi_{4s+1}, \dots, \chi_{4Q}, \dots, \chi_{4S})$. Considering the QBD structure of the Markov process, stability exists in our model if it satisfie the condition $\chi F_0 e < \chi F_2 e$. This condition is both necessary and sufficien for the stability of this queueing model under study, and it reduces to $\lambda < \mu$.

3.3. The stationary probability vector

Let X be the stationary probability vector of the infintesima generator $\mathbb{Q}$ of the process {X(t): $t \geq 0$}. The subdivision of $X = (x_0, x_1, x_2, ...)$, where x_0 is of dimension $2(S+1)m + 2Snm$ and $x_1, x_2, ...$ are of dimension $2(S+1)m + 2Snm + Slm$. As X is a vector satisfie the relation $X\mathbb{Q} = 0$ and $Xe = 1$. The probability vector X follows a matrix geometric structure under the steady state is

$$x_j = x_1 R^{j-1}, \ j \geq 2 \tag{2}$$

where R is the quadratic equation's lowest non-negative solution $R^2F_2 + RF_1 + F_0 = 0$ and the vector x_0, x_1 are obtained with the help of succeeding equations:

$$x_0A_{00} + x_1A_{10} + \sum_{i=2}^{\infty} x_iA = 0, \quad (3)$$

$$x_0A_{01} + x_1[F_1 + RF_2] = 0, \quad (4)$$

subject to a condition normalization

$$x_0e_{2(S+1)m+2Snm} + x_1[I-R]^{-1}e_{2(S+1)m+2Snm+Slm} = 1. \quad (5)$$

The rate matrix R can be computed with the help of the following iteration formula which has been suggested by Neuts [10] $R(n+1) = -F_0F_1^{-1} - R^2(n)F_2F_1^{-1}$ for $n \geq 0$ where $R(0) = 0$. Since F_1^{-1} and $(F_0 + R_2F_2)$ are positive, the rate matrix R will converge and so the entries of R will increase monotonically in the successive iterations. Iteration may be terminated when the condition $max_{i,j}[R_{ij}(n+1) - R_{ij}(n)] < e$ is attained. Here, e denotes the degree of accuracy and $R(n)$ indicates the value of the rate matrix at the n-th iteration.

4. System characteristics

- Probability that the server is idle in regular process
  $P_{INM} = \sum_{i=0}^{\infty} \sum_{u_1=0}^{S} \sum_{u_4=1}^{m} x_{i0u_1u_4}$.
- Probability that the server is idle in working vacation process
  $P_{IWV} = \sum_{i=0}^{\infty} \sum_{u_1=0}^{S} \sum_{u_4=1}^{m} x_{i2u_1u_4}$.
- Probability that the server is busy in regular process
  $P_{BNM} = \sum_{i=0}^{\infty} \sum_{u_1=1}^{S} \sum_{u_3=1}^{n} \sum_{u_4=1}^{m} x_{i1u_1u_3u_4}$.
- Probability that the server is busy in working vacation
  $P_{BWV} = \sum_{i=0}^{\infty} \sum_{u_1=1}^{S} \sum_{u_3=1}^{n} \sum_{u_4=1}^{m} x_{i3u_1u_3u_4}$.
- Probability that the server is breakdown
  $P_{BD} = \sum_{i=1}^{\infty} \sum_{u_1=1}^{S} \sum_{u_2=1}^{l} \sum_{u_4=1}^{m} x_{i4u_1u_2u_4}$.
- Expected number of customers in the orbit
  $E_{orbit} = \sum_{i=1}^{\infty} ix_ie$.
- Probability that the server is busy
  $P_{Busy} = P_{BNM} + P_{BWV}$.
- Expected number of customers in the system
  $E_{system} = E_{orbit} + P_{Busy}$.
- Expected number of items in the inventory level
  $E_{IL} = \sum_{i=0}^{\infty} \sum_{u_1=1}^{S} \sum_{u_4=1}^{m} u_1x_{i0u_1u_4} + \sum_{i=0}^{\infty} \sum_{u_1=1}^{S} \sum_{u_3=1}^{n} \sum_{u_4=1}^{m} u_1x_{i1u_1u_3u_4}$
  $+ \sum_{i=0}^{\infty} \sum_{u_1=1}^{S} \sum_{u_4=1}^{m} u_1x_{i2u_1u_4} + \sum_{i=0}^{\infty} \sum_{u_1=1}^{S} \sum_{u_3=1}^{n} \sum_{u_4=1}^{m} u_1x_{i31u_1u_2u_3u_4}$
  $+ \sum_{i=1}^{\infty} \sum_{u_1=1}^{S} \sum_{u_2=1}^{l} \sum_{u_4=1}^{m} u_1x_{i4u_1u_2u_4}$.
- Expected reorder rate
  $E_R = \sum_{i=0}^{\infty} \sum_{u_3=1}^{n} \sum_{u_4=1}^{m} x_{i1(s+1)u_3u_4}(U^0 \otimes I_m)e + \sum_{i=0}^{\infty} \sum_{u_3=1}^{n} \sum_{u_4=1}^{m} x_{i3(s+1)u_3u_4}(\theta U^0 \otimes I_m)e$.
- The effective retrial rate
  $\Delta = \delta \sum_{i=1}^{\infty} \sum_{u_1=1}^{S} \sum_{u_4=1}^{m} x_{i0u_1u_4} + \delta \sum_{i=1}^{\infty} \sum_{u_1=1}^{S} \sum_{u_4=1}^{m} x_{i2u_1u_4}$.

5. Cost Analysis

The total cost for our model is given below, with the cost elements (per unit time) related to various system measures.

$$TC = c_wE_{system} + c_hE_{IL} + c_sE_R$$

where

- TC: Total cost (per unit time)

- c_h: The inventory holding cost (per unit time)
- c_w: Waiting cost of a customer in the system (per unit time)
- c_s: Setup cost (per order)

6. Numerical Implementation

To compute numerical outcomes, we have employed diverse MAP demonstrations for the incoming arrival in a manner that ensures their mean values are 1, as recommended by [5].

- **Erlang arrival (ERA):**

$$D_0 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix} D_1 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

- **Exponential arrival (EXA):**

$$D_0 = [-1] D_1 = [1]$$

- **Hyper exponential arrival** $(HEXA)$**:**

$$D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix} D_1 = \begin{bmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{bmatrix}$$

Consider the following PH-distributions for the service and repair progression:

- **Erlang service** (ERS)**:**

$$\gamma = [1, 0] \quad U = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$$

- **Erlang repair** (ERR)**:**

$$\alpha = [1, 0] \quad T = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$$

- **Exponential service** (EXS)**:**

$$\gamma = [1] \quad U = [-1]$$

- **Exponential repair** (EXR)**:**

$$\alpha = [1] \quad T = [-1]$$

- **Hyper exponential service** $(HEXS)$**:**

$$\gamma = [0.8, 0.2] \quad U = \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}$$

- **Hyper exponential repair** $(HEXR)$**:**

$$\alpha = [0.8, 0.2] \quad T = \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}$$

Illustration 1

For this both policies, it was assumed that values of all parameters of the QIS were fixe except the service rate μ: $\lambda = 1$, $\eta = 3$, $\theta = 0.6$, $\tau = 2$, $\beta = 2$, $\psi = 1$, $\delta = 3$, $\sigma = 0.5$, $p_1 = p_2 = 0.6$, $q_1 = q_2 = 0.4$, $s = 5$, $S = 15$.
Here, we compare and analyse the two policy (s, S) and (s, Q) as follows in tables 1-6:

- First, we observe that both E_{system} and E_{orbit} in Table 1-6 under varying service rate μ , it is gradually decreases as μ increase for both (s, S) and (s, Q) but the notable is (s, S) policy give the minimum for both E_{system} and E_{orbit}.
- Observe the service times, E_{system} and E_{orbit} are decreases highly in $HEXS$ and slowly decrease in ERS than all other service times. Likewise, from the view point of arrival times, E_{system} and E_{orbit} are decreases highly for $HEXA$ compared to other arrival times.

Table 1: *Service rate (μ) vs E_{system} and E_{orbit} - ERA*

	ERS		EXS		$HEXS$	
μ	E_{system}	E_{orbit}	E_{system}	E_{orbit}	E_{system}	E_{orbit}
15	0.081396697	0.047355675	0.116583261	0.046932851	0.060209747	0.031898478
16	0.075864324	0.043882684	0.109224350	0.043648505	0.057304727	0.030565025
17	0.071064407	0.040901610	0.102739862	0.04079181	0.054674483	0.029325579
18	0.066854402	0.038311241	0.096982359	0.038284564	0.052279029	0.028171817
19	0.063128015	0.036037241	0.091835886	0.036066509	0.050086560	0.027096253
20	0.059803954	0.034023538	0.087207964	0.034090466	0.048071247	0.02609211
21	0.056818744	0.032226894	0.083023918	0.032318974	0.046211764	0.025153247
22	0.054121959	0.030613354	0.079222779	0.030721913	0.044490267	0.024274106
23	0.051672939	0.029155827	0.075754267	0.029274789	0.042891663	0.023449656
24	0.049438484	0.027832404	0.072576547	0.027957481	0.041403065	0.022675351

Table 2: *Service rate (μ) vs E_{system} and E_{orbit} - EXA*

	ERS		EXS		$HEXS$	
μ	E_{system}	E_{orbit}	E_{system}	E_{orbit}	E_{system}	E_{orbit}
15	0.093658859	0.057831180	0.125620027	0.057370051	0.077226462	0.047628434
16	0.087616380	0.053656465	0.117884279	0.053393412	0.073004243	0.044783511
17	0.082319352	0.050040468	0.111041640	0.049917853	0.069231262	0.042257589
18	0.077636049	0.046878488	0.104946594	0.046856002	0.065837924	0.039999559
19	0.073464370	0.044090388	0.099483485	0.044139417	0.062768568	0.037968788
20	0.069723889	0.041613800	0.094559255	0.041713775	0.059978093	0.036132537
21	0.066350351	0.039399467	0.090098203	0.039535447	0.057429527	0.034464111
22	0.063291771	0.037407970	0.086038129	0.037569008	0.055092251	0.032941515
23	0.060505619	0.035607397	0.082327459	0.035785419	0.052940673	0.031546455
24	0.057956750	0.033971637	0.078923079	0.034160655	0.050953225	0.030263586

Table 3: *Service rate (μ) vs E_{system} and E_{orbit} - HEXA*

	ERS		EXS		$HEXS$	
μ	E_{system}	E_{orbit}	E_{system}	E_{orbit}	E_{system}	E_{orbit}
15	0.130072755	0.085272901	0.140741030	0.072324015	0.085907558	0.047067013
16	0.118644770	0.076620377	0.131673874	0.066854199	0.080218556	0.043713552
17	0.109278270	0.069644502	0.123722961	0.062135811	0.075394726	0.040903528
18	0.101432497	0.063889862	0.116692238	0.058026255	0.071233209	0.03850289
19	0.094745436	0.059054418	0.110429349	0.054416549	0.06759258	0.036419856
20	0.088964620	0.054929433	0.104814044	0.051222021	0.064370838	0.034589272
21	0.083908013	0.051365737	0.099750091	0.048375934	0.061492291	0.032963458
22	0.079440679	0.048253720	0.095159543	0.045824997	0.058899399	0.031506611
23	0.075460231	0.045510930	0.090978563	0.043526164	0.056547500	0.030191241
24	0.071887408	0.043074081	0.087154360	0.041444300	0.054401305	0.02899583

Table 4: *Service rate (μ) vs E_{system} and E_{orbit} - ERA*

	ERS		EXS		$HEXS$	
μ	E_{system}	E_{orbit}	E_{system}	E_{orbit}	E_{system}	E_{orbit}
15	0.082004602	0.047355519	0.116584109	0.046933442	0.060824563	0.031913956
16	0.076429926	0.043882824	0.109225231	0.043649085	0.057874696	0.030578216
17	0.071593705	0.040901916	0.102740765	0.040792375	0.055206267	0.029336964
18	0.067352119	0.038311641	0.096983274	0.038285111	0.052777843	0.028181749
19	0.063597952	0.036037694	0.091836807	0.036067038	0.050556568	0.027104999
20	0.060249224	0.034024019	0.087208889	0.034090977	0.048515832	0.026099873
21	0.057241938	0.032227388	0.083024844	0.032319470	0.046633719	0.025160188
22	0.054525258	0.030613851	0.079223704	0.030722393	0.044891928	0.024280350
23	0.052058203	0.029156322	0.075755191	0.029275254	0.043275004	0.023455306
24	0.049807313	0.027832894	0.072577470	0.027957933	0.041769774	0.022680491

Table 5: *Service rate (μ) vs E_{system} and E_{orbit} - EXA*

	ERS		*EXS*		*HEXS*	
μ	E_{system}	E_{orbit}	E_{system}	E_{orbit}	E_{system}	E_{orbit}
15	0.094342828	0.057831783	0.125622511	0.057371230	0.077912574	0.047638686
16	0.088262387	0.053657353	0.117886912	0.053394652	0.073653585	0.044793161
17	0.082931496	0.050041572	0.111044412	0.049919149	0.069847634	0.042266700
18	0.078217783	0.046879757	0.104949496	0.046857350	0.066424553	0.040008185
19	0.074018636	0.044091787	0.099486509	0.044140813	0.063328223	0.037976979
20	0.070253213	0.041615303	0.094562394	0.041715216	0.060513165	0.036140334
21	0.066856921	0.039401053	0.090101450	0.039536929	0.057942099	0.034471552
22	0.063777495	0.037409625	0.086041476	0.037570530	0.055584149	0.032948633
23	0.060972173	0.035609109	0.082330902	0.035786976	0.053413507	0.031553280
24	0.058405612	0.033973398	0.078926612	0.034162246	0.051408424	0.030270142

Table 6: *Service rate (μ) vs E_{system} and E_{orbit} - HEXA*

	ERS		*EXS*		*HEXS*	
μ	E_{system}	E_{orbit}	E_{system}	E_{orbit}	E_{system}	E_{orbit}
15	0.131245994	0.085194321	0.14075261	0.072332391	0.087390312	0.047182656
16	0.119735103	0.076569695	0.131688802	0.066864727	0.081577421	0.043823642
17	0.110296386	0.069612267	0.123740628	0.062148038	0.076649406	0.041007718
18	0.102387591	0.063870343	0.116712188	0.058039850	0.072399139	0.03860134
19	0.09564524	0.059043966	0.110451236	0.054431263	0.068682030	0.036512935
20	0.089815602	0.054925634	0.104837597	0.051237669	0.065393700	0.034677418
21	0.084715599	0.051366937	0.099775098	0.048392369	0.062456649	0.033047113
22	0.080209444	0.048258756	0.095185831	0.045842105	0.059811936	0.031586194
23	0.07619406	0.045518963	0.091005993	0.043543856	0.057413806	0.030267136
24	0.072589624	0.043084494	0.087182817	0.041462502	0.055226098	0.029068385

Illustration 2

We picturise the consequences of the breakdown rate ψ against the P_{busy}. Fix $\lambda = 1$, $\mu = 15$, $\theta = 0.6$, $\eta = 3$, $\tau = 5$, $\beta = 2$, $\delta = 3$, $\sigma = 0.5$, $p_1 = p_2 = 0.6$, $q_1 = q_2 = 0.4$, $s = 5$, $S = 15$, these values satisfy the condition for stability. From the figu es 2 - 4: we can explore that while increasing the server's breakdown rate (ψ), P_{busy} decreases for all feasible provisions of incoming arrival and service patterns. As increase in breakdown rate indicates that customers will frequently be unable to access the server, which is decreases of P_{busy} is higher for $HEXA$ and lower for ERA. Likewise, it is higher for ERS and lower for $HEXS$.

Illustration 3

To investigate the impact of the TC on both the service (μ) and repair (τ) rates in the Figures 5-13. Fix $\lambda = 1$, $\sigma = 0.2$, $\theta = 0.6$, $\beta = 3$, $\delta = 3$, $p_1 = p_2 = 0.6$, $q_1 = q_2 = 0.4$, $s = 5$, $S = 15$, $C_H = 70$, $C_I = 110$, $C_R = 120$, such that the system leftovers stable.
From the viewpoint of Figures 5-13, we maximize both the service and repair rates for all possible groups of arrival and service times, we notice that the TC decreases. Consider the service times, TC decreases exceedingly for ERS and decreases moderately for EXS. Therefore, TC decreases slowly for ERA and rapidly for $HEXA$.

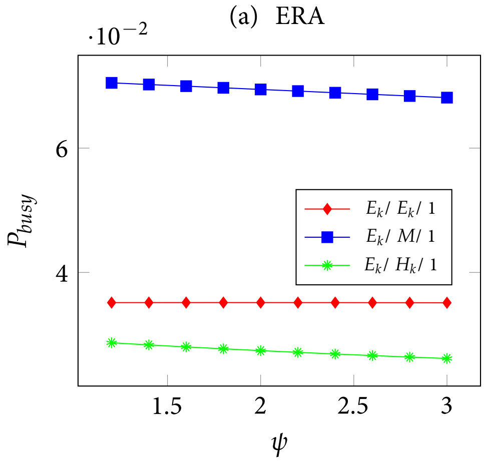

Figure 2: *Breakdown rate vs. P_{busy}*

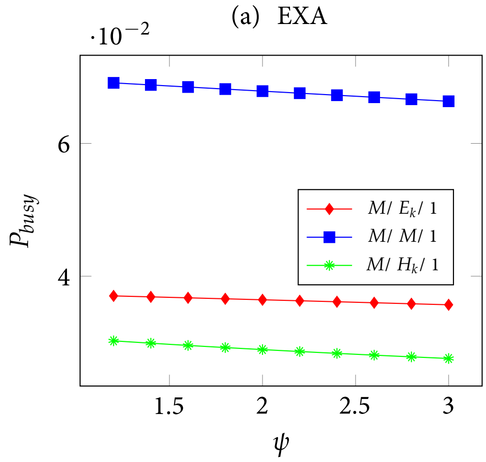

Figure 3: *Breakdown rate vs. P_{busy}*

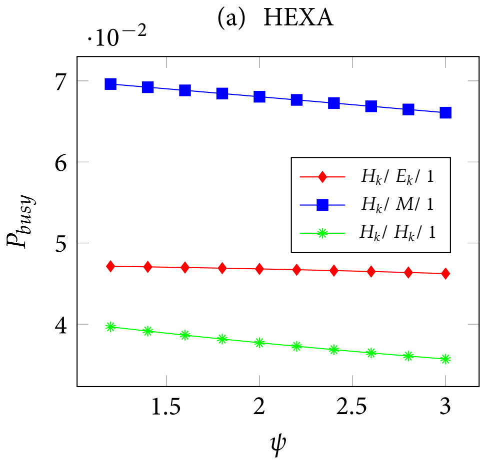

Figure 4: *Breakdown rate vs. P_{busy}*

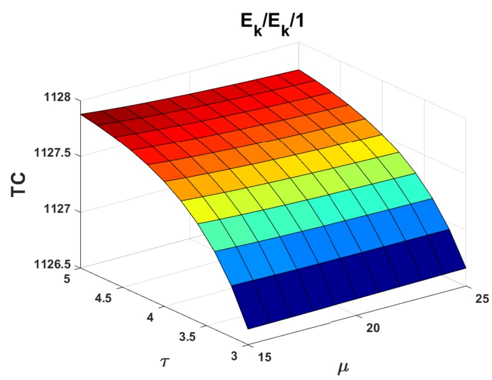

Figure 5: *Service and repair rates vs. TC*

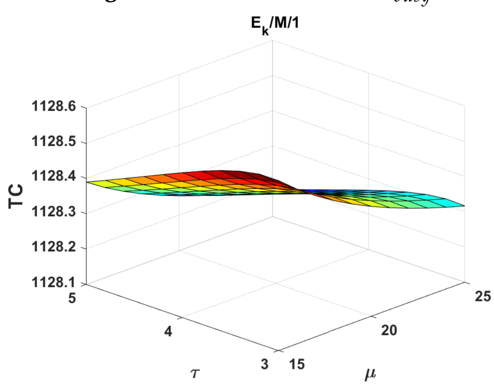

Figure 6: *Service and repair rates vs. TC*

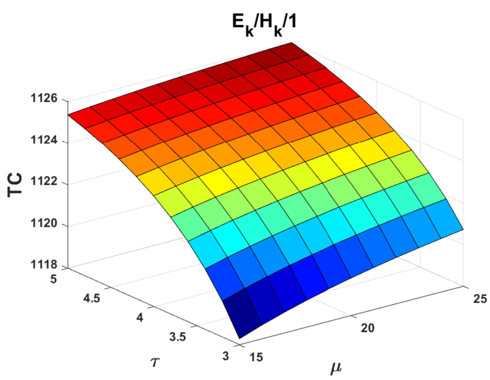

Figure 7: *Service and repair rates vs. TC*

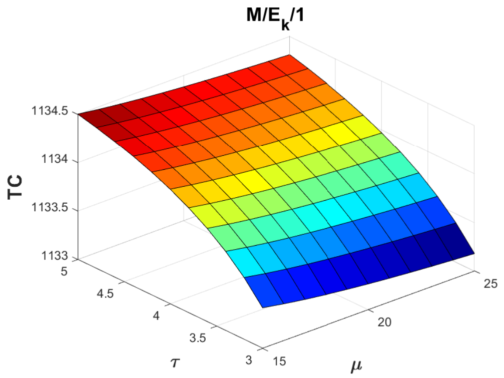

Figure 8: *Service and repair rates vs. TC*

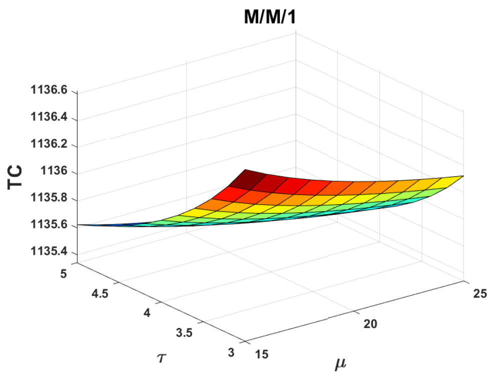

Figure 9: *Service and repair rates vs. TC*

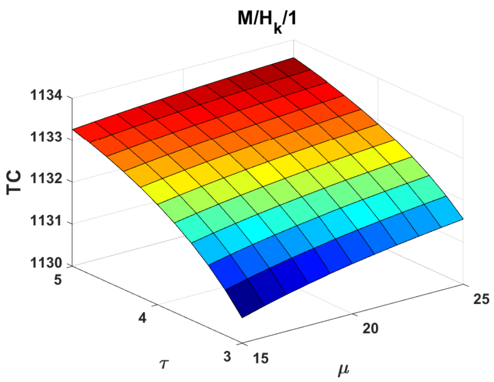

Figure 10: *Service and repair rates vs. TC*

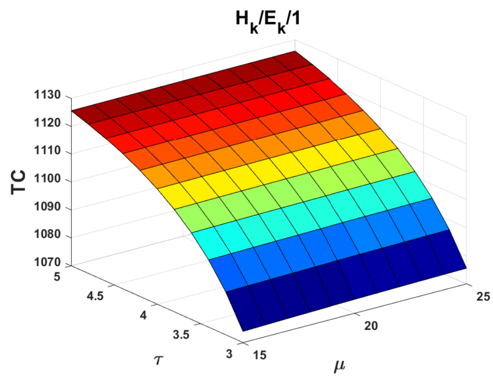

Figure 11: *Service and repair rates vs. TC*

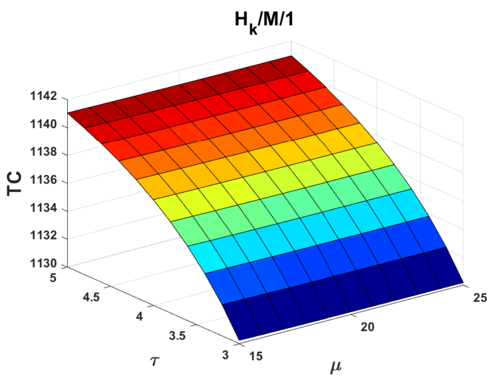

Figure 12: *Service and repair rates vs. TC*

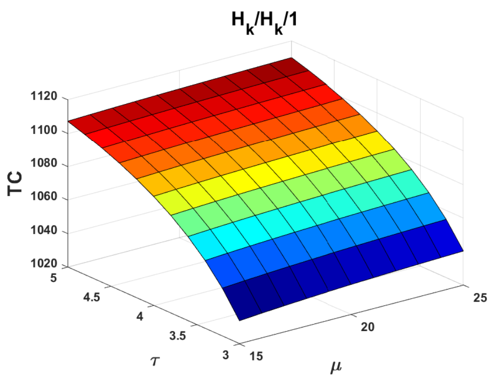

Figure 13: *Service and repair rates vs. TC*

7. Conclusion

A retrial inventory model with MAP arrivals, PH-distributed service, working vacations, collision of orbital customers, flus out, balking, breakdown and repair has been investigated. The peculiarity of this model is that the server can offer service even in the vacation period and the system is always stable because of the flus out of the system. We have considered MAP for arrivals and would like to extend our models by considering BMAP for arrivals which is best suited for modelling arrivals which come in batches.

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