MULTICOMPONENT RELIABILITY UNDER PATHWAY MODEL

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Abstract

In this paper, we consider a system with a finite number of components. It is assumed that the system architecture is a series format. The system fails when any one of the components fails. The case where the lifetimes of the components, are independently distributed and have pathway density is considered. Then the survival function, hazard function, the expected time to failure, general moments, etc. of the system lifetime are computed. It is shown that the hazard function can have many types of shapes, including bathtub shapes. The estimation of stress-strength reliability is considered based on the method of maximum likelihood estimation when both stress and strength variables follow the pathway model. Finally, to show the applicability of the proposed model in a real-life scenario, remission time data from cancer patients is analyzed.

Keywords: Survival Function, Pathway Distribution, Multicomponent Reliability, Stress-Strength Reliability, Expected Time to Failure

1. Introduction

Consider a multicomponent system consisting of k components, connected in a series format so that the system fails if any of the k components fails. Let the lifetimes of the components be the random variables $X_1, ..., X_k$, $X_j > 0, j = 1, ..., k$. Let X be the minimum, $X = \min\{X_1, ..., X_k\}$. Then the system failure time is X. Suppose that the components are functioning independently. Then the probability that $X > t$ for some t is given by

$$Pr\{X > t\} = Pr\{X_1 > t\}Pr\{X_2 > t\}...Pr\{X_k > t\}. \quad (1)$$

That is, in terms of the distribution functions

$$1 - F_X(t) = [1 - F_{X_1}(t)]...[1 - F_{X_k}(t)]$$

where $F_{X_j}(t) = Pr\{X_j \leq t\}$ is the distribution function of $X_j, j = 1, ..., k$ and $F_X(t)$ is that of X. Then the density of X, if $F_{X_j}(t)$ is differentiable, is given by the following:

$$f_X(t) = -\frac{\mathrm{d}}{\mathrm{d}t}[1 - F_X(t)] = \sum_{j=1}^{k} \left\{ [-\frac{\mathrm{d}}{\mathrm{d}t} Pr\{X_j > t\}] \prod_{i \neq j=1}^{k} Pr\{X_i > t\} \right\}. \quad (2)$$

Basic notions of reliability analysis may be seen, from [1], [2], and [3]. Reliability analysis for dependent cases may be seen, for example, from [4], [5] and [6]. We will examine (2) and study its properties and connections to various problems in different fields. First, we will consider the case when the density of x_j belongs to the general family of functions called the pathway model. The original pathway model was introduced by Mathai [7] for the real rectangular matrix-variate case. Later, Mathai and Provost [8] was extended it to the complex domain. The pathway model for the real scalar positive variable case can be stated as follows:

$$f_1(x) = c_1 x^{\gamma}[1 - a(1-q)x^{\delta}]^{\frac{\eta}{1-q}}, q < 1 \tag{3}$$

for $a > 0, \delta > 0, \eta > 0, \gamma > -1, 1 - a(1-q)x^{\delta} > 0$ and $f_1(x) = 0$ elsewhere. The functional part of the basic type-1 beta density is $x^{\alpha-1}(1-x)^{\beta-1}, 0 \leq x \leq 1, \alpha > 0, \beta > 0$ and zero elsewhere. Hence (3) can be looked upon as a generalized type-1 beta form, that is, for $\frac{\eta}{1-q} = \beta - 1, \delta = 1, q = 0, \gamma = \alpha - 1$ one has the type-1 beta form. Note that one can also relocate the variable x. Write the model as

$$f_2(x) = c_2(x-\alpha)^{\gamma}[\beta - a(1-q)(x-\alpha)^{\delta}]^{\frac{\eta}{1-q}}, q < 1 \tag{4}$$

for $\eta > 0, a > 0, \delta > 0, x \geq \alpha, \beta > 0, \alpha > 0, 0 < \alpha \leq x \leq \alpha + [\frac{\beta}{a(1-q)}]^{\frac{1}{\delta}}$. Note that the basic type-1 beta model, triangular density, power function model, uniform density, etc are particular cases of (3). The limiting form of the exponentiated versions of (3) and (4) can also be shown to be Bose-Einstein density in Physics. Note that when q approaches 1 then the support will extend to $0 \leq x < \infty$ from the finite range support in (3). For $q > 1$, write $1 - q = -(q-1)$ so that the model in (3) switches into the model, which is another family of functions,

$$f_3(x) = c_3 x^{\gamma}[1 + a(q-1)x^{\delta}]^{-\frac{\eta}{q-1}}, q > 1 \tag{5}$$

for $a > 0, \eta > 0, \gamma > -1, \delta > 0, x \geq 0$. The functional part of the basic type-2 beta density is $x^{\alpha-1}(1+x)^{-(\alpha+\beta)}, 0 \leq x < \infty, \alpha > 0, \beta > 0$. Hence (5) can be looked upon as a generalized type-2 beta model. If relocation of the variable is required, then replace x in (5) by $x - \alpha > 0$ so that $0 < \alpha \leq x < \infty$. Observe that the standard F-density, type-2 beta density, Pareto density, etc are particular cases in (5). The exponentiated version of (5), that is, put $x = \mathrm{e}^{-cy}, c > 0, -\infty < y < \infty$ is connected to various densities such as the generalized logistic density, see [9], the standard logistic density, a limiting form giving rise to the famous Fermi-Dirac density in Physics also. Now, let $q \to 1_-$ in (3) and $q \to 1_+$ in (5). Then both the models in (3) and (5) go to

$$f_4(x) = c_4 x^{\gamma} \mathrm{e}^{-a\eta x^{\delta}}, a > 0, \eta > 0, \delta > 0, x \geq 0 \tag{6}$$

and zero elsewhere. We may also relocate the variable, if necessary. Observe that (6) is in the form of a generalized gamma density. For $\gamma = \delta - 1$ it is the Weibull density. The standard gamma density, chisqure density, exponential, density, Maxwell-Boltzmann density, Raleigh density, etc are special cases of (6). Thus, (3) or (5) is the basic pathway model or all cases of (3), (5) and (6) are contained in (3) or (5). For $q < 1, q > 1, q \to 1$ will cover almost all densities in current use and all these are contained in (3) or (5). Hence a wide spectrum of models is covered in the problems that we discuss in this paper. The advantage of the model in (3) or (5) in a model building situation is the following: If $f_1(x), f_3(x), f_4(x)$ are to be treated as statistical densities, then c_1, c_2, c_3 are the normalizing constants, there and they are the following:

$$c_1 = \frac{\delta[a(1-q)]^{\frac{\gamma+1}{\delta}}\Gamma(\frac{\eta}{1-q} + 1 + \frac{\gamma+1}{\delta})}{\Gamma(\frac{\gamma+1}{\delta})\Gamma(\frac{\eta}{1-q} + 1)}, q < 1, \gamma + 1 > 0, a, \delta, \eta > 0 \tag{7}$$

.

$$c_3 = \frac{\delta[a(q-1)]^{\frac{\gamma+1}{\delta}}\Gamma(\frac{\eta}{q-1})}{\Gamma(\frac{\gamma+1}{\delta})\Gamma(\frac{\eta}{q-1} - \frac{\gamma+1}{\delta})} q > 1, \gamma + 1, a, \delta, \eta > 0, \frac{\eta}{q-1} - \frac{\gamma+1}{\delta} > 0 \tag{8}$$

.

$$c_4 = \frac{\delta(a\eta)^{\frac{\gamma+1}{\delta}}}{\Gamma(\frac{\gamma+1}{\delta})}, \gamma+1>0, \delta>0, a>0, \eta>0. \tag{9}$$

For $\delta = 1, a = 1, \eta = 1, \gamma = 0$ in (3) gives the famous Tsallis statistics in nonextensive statistical mechanics. This Tsallis statistic is valid for $q < 1, q > 1, q \to 1$ situations. It is stated that over 3000 articles were written on this Tsallis statistics between 1990 and 2010 period. Tsallis statistics, excluding the normalizing constant, is a power function model in the sense

$$\frac{\mathrm{d}}{\mathrm{d}x} f_1(x) = -[f_1(x)]^q.$$

For $a = 1, \delta = 1, \eta = 1$, (5) gives superstatistics in statistical mechanics. This is valid for $q > 1, q \to 1$ situations but not for $q < 1$. Dozens of articles are also published in this area. The development in Tsallis statistics is available from his book, see [10]. The basic paper on superstatistics is by [11]. From a physical point of view, superstatistics is constructed by superimposing a distribution over another distribution. But from a statistical point of view, superstatistics is nothing but an unconditional density in a Bayesian setup when both the conditional density and prior density belong to generalized gamma families. By using this pathway model several compound distributions are developed for details see, [12],[13] and [14].

1.1. A particular case

Our interest here is to examine the multi-component system failure under a pathway model of (3), thereby (5) and (6) for the particular case $\gamma = \delta - 1$. In this case, the normalizing constants simplify and the models go into very simple forms. This particular case of (3),(5) and (6) is the following:

$$f_5(x) = a\delta(\eta+1-q)x^{\delta-1}[1-a(1-q)x^{\delta}]^{\frac{\eta}{1-q}}, q<1 \tag{10}$$

for $a > 0, \delta > 0, \eta > 0, \eta + 1 - q > 0, 1 - a(1-q)x^{\delta} > 0$.

$$f_6(x) = a\delta(\eta+1-q)x^{\delta-1}[1+a(q-1)x^{\delta}]^{-\frac{\eta}{q-1}}, q>1 \tag{11}$$

for $a > 0, \delta > 0, \eta > 0, \eta + 1 - q > 0, x \geq 0$.

$$f_7(x) = a\delta\eta x^{\delta-1}\mathrm{e}^{-a\eta x^{\delta}}, \delta>0, a>0, \eta>0. \tag{12}$$

The corresponding survival probabilities are the following:

$$S_5(t) = [1-a(1-q)t^{\delta}]^{\frac{\eta}{1-q}+1}, q<1, \tag{13}$$

for $a > 0, \delta > 0, \eta > 0, 1 - a(1-q)t^{\delta} > 0$.

$$S_6(t) = [1+a(q-1)t^{\delta}]^{-\frac{\eta}{q-1}+1}, q>1, a, \delta, \eta>0, t\geq 0 \tag{14}$$

.

$$S_7(t) = \mathrm{e}^{-a\eta t^{\delta}}, a>0, \eta>0, t\geq 0. \tag{15}$$

2. Multicomponet Failure Under Pathway Model

Recall the probability of failure from (1). Then, under the pathway model of (4) to (6) for the particular case $\gamma = \delta - 1$ it is the following, writing for convenience the form in (14) for $q > 1$:

$$Pr\{x > t\} = \prod_{j=1}^{k}[1+a_j(q_j-1)t^{\delta_j}]^{-\frac{\eta_j}{q_j-1}+1} \tag{16}$$

for $a_j > 0, \delta_j > 0, \eta_j > 0, \eta_j + 1 - q_j > 0, q_j > 1, q_j < 1, q_j \to 1, j = 1, ..., k$. For any particular j, we can take the form in (13) or (14) or (15). Thus, (16) gives a very rich family of probabilities. The density of x in this case, denoted by $f(x)$, is the following:

$$\begin{aligned} f(t) &= -\frac{\mathrm{d}}{\mathrm{d}t} Pr\{x > t\} = \sum_{j=1}^{k} (\eta_j + 1 - q_j) a_j \delta_j t^{\delta_j - 1} \\ &\times \; [1 + a_j(q_j - 1)t^{\delta_j}]^{-\frac{\eta_j}{q_j - 1}} \{ \prod_{i \neq j = 1}^{k} [1 + a_i(q_i - 1)t^{\delta_i}]^{-\frac{\eta_i}{q_i - 1} + 1} \}. \end{aligned} \tag{17}$$

Therefore the hazard function of x, denoted by $h(t)$, is the following:

$$h(t) = \frac{f_x(t)}{Pr\{x > t\}} = \sum_{j=1}^{k} \frac{(\eta_j + 1 - q_j) a_j \delta_j t^{\delta_j - 1}}{1 + a_j(q_j - 1)t^{\delta_j}} \tag{18}$$

for $q_j > 1, q_j < 1, q_j \to 1, \eta_j > 0, \eta_j + 1 - q_j > 0, a_j > 0, \delta_j > 0, j = 1, ..., k$. Observe that when a $q_j \to 1$ for a particular j, the corresponding term is simply $\eta_j a_j \delta_j t^{\delta_j - 1}$. Thus, a rich variety of hazard functions of having curves of various shapes are available from (18). For example, for $k = 2, q_2 \to 1$ we have the form, denoted by $h(t)$,

$$h(t) = \frac{(\eta_1 + 1 - q_1) a_1 \delta_1 t^{\delta_1 - 1}}{1 + a_1(q_1 - 1)t^{\delta_1}} + \eta_2 a_2 \delta_2 t^{\delta_2 - 1}, q_1 > 1. \tag{19}$$

The different shapes of the hazard function of multicomponent systems under the pathway model are demonstrated.

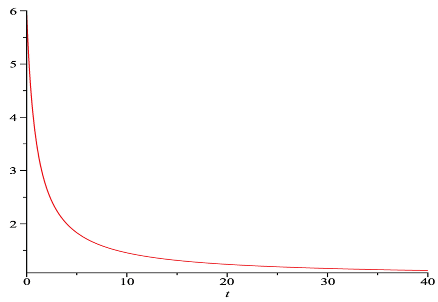

Figure 1

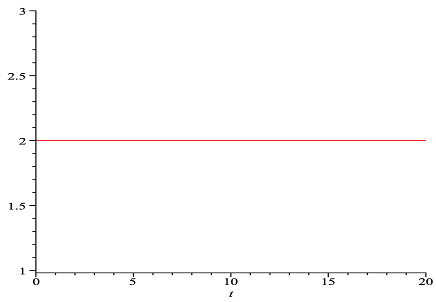

Figure 2

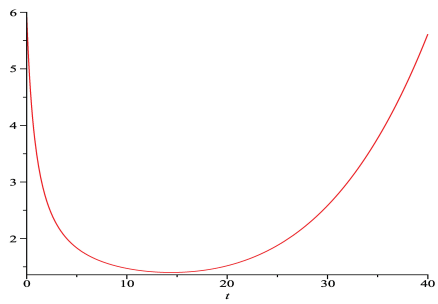

Figure 3

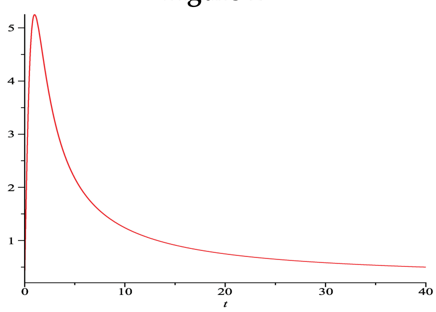

Figure 4

Figure 1: $\eta_1 = 3, \eta_2 = 1, q_1 = 1.5, a_1 = 2, a_2 = 1, \delta_1 = 1, \delta_2 = 1$
Figure 2: $\eta_1 = 0.5, \eta_2 = 2, q_1 = 1.5, a_1 = 2, a_2 = 1, \delta_1 = 1, \delta_2 = 1$
Figure 3: $\eta_1 = 3, \eta_2 = \frac{1}{1500}, q_1 = 1.5, a_1 = 2, a_2 = \frac{1}{2}, \delta_1 = 1, \delta_2 = 1$
Figure 4: $\eta_1 = 3, \eta_2 = \frac{1}{2}, q_1 = 1.5, a_1 = 2, a_2 = \frac{1}{2}, \delta_1 = 1, \delta_2 = 1$

Another case for $k = 2$, $q_1 > 1$ and $q_2 < 1$, $h(t)$ becomes;

$$h(t) = \frac{(\eta_1 + 1 - q_1) a_1 \delta_1 t^{\delta_1 - 1}}{1 + a_1(q_1 - 1)t^{\delta_1}} + \frac{(\eta_2 + 1 - q_2) a_2 \delta_2 t^{\delta_2 - 1}}{1 - a_2(1 - q_2)t^{\delta_2}}, \tag{20}$$

for $q_1 > 1, q_2 < 1, a_j > 0, \delta_j > 0, \eta_j > 0, \eta_j + 1 - q_j > 0, j = 1, 2$. All types of shapes are available from (20).

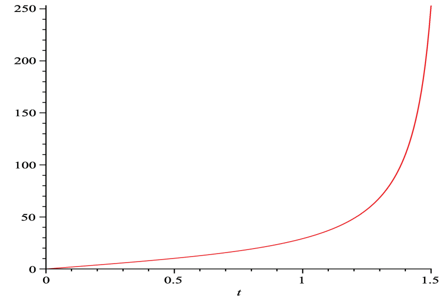

Figure 5

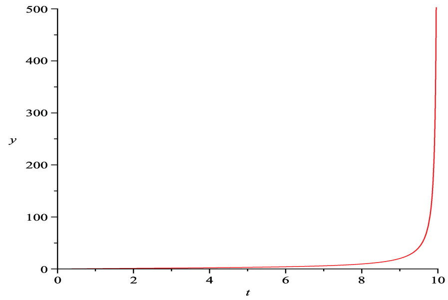

Figure 6

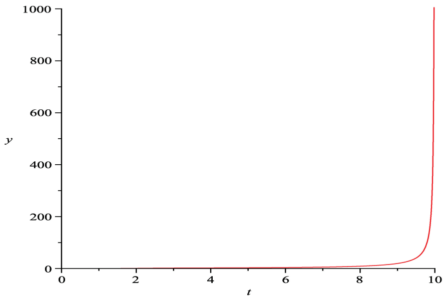

Figure 7

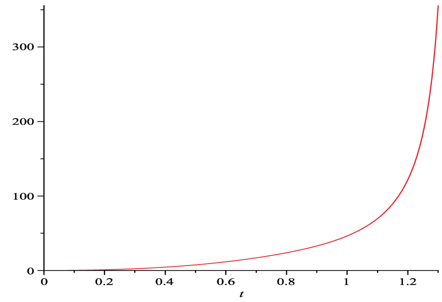

Figure 8

Figure 5: $\eta_1 = 2, \eta_2 = 2, q_1 = 1.9, q_2 = 0.9, a_1 = 1, a_2 = 4, \delta_1 = 2, \delta_2 = 2$
Figure 6: $\eta_1 = 2, \eta_2 = 2, q_1 = 1.9, q_2 = 0.9, a_1 = \frac{1}{10}, a_2 = \frac{1}{10}, \delta_1 = 2, \delta_2 = 2$
Figure 7: $\eta_1 = 2, \eta_2 = 2, q_1 = 1.9, q_2 = 0.9, a_1 = \frac{1}{100}, a_2 = \frac{1}{100}, \delta_1 = 5, \delta_2 = 3$
Figure 8: $\eta_1 = 2, \eta_2 = 2, q_1 = 1.9, q_2 = 0.9, a_1 = 3, a_2 = 4, \delta_1 = 5, \delta_2 = 3$

2.1. Expected time to failure

From here onward, all discussions connected with $k = 2$ also contain the case of $k - 1$ of the lifetimes $x_1, \ldots, x_k$ are identically distributed so that there will only be two distinct densities. This can be computed from the density of x itself or from the survival function of x. That is,

$$E(t) = \int_0^\infty t f_x(t)\mathrm{d}t = \int_0^\infty S_x(t)\mathrm{d}t \tag{21}$$

where $S_x(t) = Pr\{x > t\}$ is the survival function of t. Integration by parts once gives the second part in (21). Hence the ρ-th moment of the time to failure is the following:

$$\begin{aligned} E(t^\rho) &= \int_0^\infty t^\rho f_x(t)\mathrm{d}t = \rho \int_0^\infty t^{\rho-1} S_x(t)\mathrm{d}t \\ &= \rho \int_0^\infty t^{\rho-1} \{\prod_{j=1}^{k} [1 + a_j(q_j - 1)t^{\delta_j}]^{-\frac{\eta_j}{q_j - 1} + 1}\}\mathrm{d}t, q_j > 1. \end{aligned} \tag{22}$$

Take $q_j < 1$ for the type-1 case and $q_j \to 1$ for the gamma case. Hence all different forms are there in (22). For $k = 2$, a general integral in this category, denoted by I_1, is the following:

$$I_1 = \int_0^\infty t^\xi [1 + a_1(q_1 - 1)t^{\delta_1}]^{-\frac{\eta_1}{q_1 - 1} + 1} [1 + a_2(q_2 - 1)t^{\delta_2}]^{-\frac{\eta_2}{q_2 - 1} + 1}\mathrm{d}t. \tag{23}$$

Replace ξ by $\rho - 1$ and multiply the integral by ρ to obtain the ρ-th moment from I_1. The integral in (23) has the structure of a Mellin convolution of a ratio. For two functions $g_1(x_1)$ and $g_2(x_2)$

the Mellin convolution of a ratio has the format

$$g(u) = \int_v v g_1(uv) g_2(v) \mathrm{d}v \tag{24}$$

so that the Mellin transform of $g(u)$, with Mellin parameter s, or

$$M_g(s) = \int_0^\infty u^{s-1} g(u) \mathrm{d}u$$

has the form

$$M_g(s) = M_{g_1}(s) M_{g_2}(2-s) \tag{25}$$

where

$$M_{g_1}(s) = \int_0^\infty x_1^{s-1} g_1(x_1) \mathrm{d} \text{ and } M_{g_2}(2-s) = \int_0^\infty x_2^{-s+1} g_2(x_2) \mathrm{d}x_2$$

where g_1 and g_2 need not be statistical densities. If they are statistical densities then the situation is the following: $M_{g_1}(s) = E(x_1^{s-1}), M_{g_2}(2-s) = E(x_2^{-s+1})$ and $u = \frac{x_1}{x_2}, x_2 = v, x_1 = uv$ and the Jacobian is v. $E[\frac{x_1}{x_2}]^{s-1} = E[x_1^{s-1}]E[x_2^{-s+1}]$ when $x_1 > 0$ and $x_2 > 0$ are independently distributed real scalar positive random variables, where E denotes the expected value. Then the density of u, denoted by $g(u)$, is available from the inverse Mellin transform. That is,

$$g(u) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} [E(u^{s-1})] u^{-s} \mathrm{d}s = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M_{g_1}(s) M_{g_2}(2-s) u^{-s} \mathrm{d}s, i = \sqrt{-1}. \tag{26}$$

In (25), g_1 and g_2 need not be statistical densities. The only condition is that the Mellin transforms exist. For the existence of inverse Mellin transform, general conditions are available, see books on complex analysis, or see [15]. For evaluating (23) let

$$g_1(x_1) = [1 + x_1^{\delta_1}]^{-\frac{\eta_1}{q_1-1}+1} \text{ and } g_2(x_2) = x_2^{\xi-1}[1 + a_2(q_2-1)x_2^{\delta_2}]^{-\frac{\eta_2}{q_2-1}+1}$$

so that for $u = [a_1(q_1-1)]^{\frac{1}{\delta_1}}$

$$\begin{aligned} g(u) &= \int_v v g_1(uv) g_2(v) \mathrm{d}v = \int_0^\infty v^\xi [1 + a_1(q_1-1)v^{\delta_1}]^{-\frac{\eta_1}{q_1-1}+1} \\ &\times [1 + a_2(q_2-1)v^{\delta_2}]^{-\frac{\eta_2}{q_2-1}+1} \mathrm{d}v = I_1 \end{aligned} \tag{27}$$

which is the right side of (23) or the item to be evaluated. But

$$\begin{aligned} M_{g_1}(s) &= \int_0^\infty x_1^{s-1} [1 + x_1^{\delta_1}]^{-\frac{\eta_1}{q_1-1}+1} \mathrm{d}x_1 \\ &= \frac{\Gamma(\frac{s}{\delta_1})\Gamma(\frac{\eta_1}{q_1-1} - 1 - \frac{s}{\delta_1})}{\delta_1 \Gamma(\frac{\eta_1}{q_1-1} - 1)} \end{aligned} \tag{28}$$

for $\eta_1 + 1 - q_1 > 0, \delta_1 > 0, \eta_1 > 0, 1 < q_1 < \eta_1 + 1, \Re(s) > 0$ where $\Re(\cdot)$ means the real part of $(\cdot)$.

$$\begin{aligned} M_{g_2}(2-s) &= \int_0^\infty x_2^{-s+1} x_2^{\xi-1} [1 + a_2(q_2-1)x_2^{\delta_2}]^{-\frac{\eta_2}{q_2-1}+1} \mathrm{d}x_2 \\ &= \frac{\Gamma(\frac{\xi-s+1}{\delta_2})\Gamma(\frac{\eta_2}{q_2-1} - 1 - \frac{(\xi-s+1)}{\delta_2})}{\delta_2 [a_2(q_2-1)]^{\frac{\xi-s+1}{\delta_2}} \Gamma(\frac{\eta_2}{q_2-1} - 1)} \end{aligned} \tag{29}$$

for $\Re(\xi - s + 1) > 0, \delta_2 > 0, \eta_2 > 0, \eta_2 + 1 - q_2 > 0, 1 < q_2 < \eta_2 + 1, \Re(\frac{\eta_2}{q_2-1} - 1 - \frac{(\xi-s+1)}{\delta_2}) > 0$.
Hence I_1 is available from the inverse Mellin transform, remembering that $u = [a_1(q_1 - 1)]^{\frac{1}{\delta_1}}$.

$$\begin{aligned} I_1 &= [\delta_1\delta_2[a_2(q_2-1)]^{\frac{\xi+1}{\delta_2}}]^{-1}[\Gamma(\frac{\eta_1}{q_1-1}-1)\Gamma(\frac{\eta_2}{q_2-1}-1)]^{-1} \\ &\times \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\frac{s}{\delta_1})\Gamma(\frac{\eta_2}{q_2-1}-1-\frac{(\xi+1)}{\delta_2}+\frac{s}{\delta_2}) \\ &\times \Gamma(\frac{\eta_1}{q_1-1}-1-\frac{s}{\delta_1})\Gamma(\frac{\xi+1}{\delta_2}-\frac{s}{\delta_2})\left[\frac{[a_1(q_1-1)]^{\frac{1}{\delta_1}}}{[a_2(q_2-1)]^{\frac{1}{\delta_2}}}\right]^{-s}\mathrm{d}s \end{aligned} \quad (30)$$

for $\max\{0, \frac{\xi+1}{\delta_2} + 1 - \frac{\eta_2}{q_2-1}\} < c < \min\{\xi + 1, \frac{\eta_1\delta_1}{q_1-1} - \delta_1\}$, $\delta_j > 0, a_j > 0, 1 < q_j < \eta_j + 1, \xi > -1, \eta_j > 0, \eta_j + 1 - q_j > 0, j = 1, 2$. This I_1 can be written as a H-function, see [16]. That is, denoting the constant part by C, we have

$$I_1 = C\, H_{2,2}^{2,2}\left[\omega\Big|_{(0,\frac{1}{\delta_1}),(\frac{\eta_2}{q_2-1}-1-\frac{(\xi+1)}{\delta_2},\frac{1}{\delta_2})}^{(2-\frac{\eta_1}{q_1-1},\frac{1}{\delta_1}),(1-\frac{(\xi+1)}{\delta_2},\frac{1}{\delta_2})}\right] \quad (31)$$

for $0 < |\omega| < 1$ where

$$\omega = \frac{[a_1(q_1-1)]^{\frac{1}{\delta_1}}}{[a_2(q_2-1)]^{\frac{1}{\delta_2}}} \text{ and } [\delta_1\delta_2[a_2(q_2-1)]^{\frac{\xi+1}{\delta_2}}]^{-1}[\Gamma(\frac{\eta_1}{q_1-1}-1)\Gamma(\frac{\eta_2}{q_2-1}-1)].$$

Observe that the roles of $[a_1(q_1 - 1)]^{\frac{1}{\delta_1}}$ and $[a_2(q_2 - 1)]^{\frac{1}{\delta_2}}$ can be interchanged by interchanging the roles of g_1 and g_2. For the existence conditions and properties of H-function see Mathai et al. (2010)[16]. MATHEMATICA programs are available for computing H-functions.
Note that I_1 of (23) has nine different forms there. We can have $q_1 > 1, (q_2 > 1, q_2 < 1, q_2 \to 1)$. Similarly for $q_1 < 1$ and $q_1 \to 1$ cases. When $q_1 \to 1$ and $q_2 \to 1$ we have the integral in (23) as

$$= \int_0^\infty t^\xi e^{-a_1\eta_1 t^{\delta_1} - a_2\eta_2 t^{\delta_2}}\mathrm{d}t. \quad (32)$$

This (32) for either $\delta_1 = 1$ or $\delta_2 = 1$ corresponds to the Laplace transform or moment-generating function of a generalized gamma density. One can go through the steps (23) to (31) and obtain the following result for $q_1 \to 1, q_2 \to 1$:

$$I_2 = [\delta_1\delta_2(a_2\eta_2)^{\frac{\xi+1}{\delta_2}}]^{-1}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\frac{s}{\delta_1})\Gamma(\frac{\xi+1-s}{\delta_2})[\frac{(a_1\eta_1)^{\frac{1}{\delta_1}}}{(a_2\eta_2)^{\frac{1}{\delta_2}}}]^{-s}\mathrm{d}s \quad (33)$$

$$= [\delta_1\delta_2(a_2\eta_2)^{\frac{\xi+1}{\delta_2}}]^{-1}H_{1,1}^{1,1}\left[\frac{(a_1\eta_1)^{\frac{1}{\delta_1}}}{(a_2\eta_2)^{\frac{1}{\delta_2}}}\Big|_{(0,\frac{1}{\delta_1})}^{(1-\frac{\xi+1}{\delta_2},\frac{1}{\delta_2})}\right], \quad (34)$$

for $\frac{(a_1\eta_1)^{\frac{1}{\delta_1}}}{(a_2\eta_2)^{\frac{1}{\delta_2}}} < 1$. The Mellin-Barnes representation in (33) can be written in the following form by replacing $\frac{s}{\delta_1}$ by s and writing $c^* = [\delta_2(a_2\eta_2)^{\frac{\xi+1}{\delta_2}}]^{-1}$:

$$I_2 = c^*\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(\frac{\xi+1}{\delta_2}-\frac{\delta_1}{\delta_2}s)[\frac{a_1\eta_1}{(a_2\eta_2)^{\frac{\delta_1}{\delta_2}}}]^{-s}\mathrm{d}s. \quad (35)$$

Evaluating this at the poles of $\Gamma(s)$ at $s = 0, -1, -2, \ldots$ the residue at $s = -\nu$ is given by

$$\lim_{s\to-\nu}(s+\nu)\Gamma(s)\Gamma(\frac{\xi+1}{\delta_2}-\frac{\delta_1}{\delta_2}s)\omega^{-s} = \frac{(-1)^\nu}{\nu!}\Gamma(\frac{\xi+1}{\delta_2}+\frac{\delta_1}{\delta_2}\nu)\omega^\nu, \omega = \frac{(a_1\eta_1)}{(a_2\eta_2)^{\frac{\delta_1}{\delta_2}}}.$$

Therefore I_2 is available as the sum of the residues.

$$I_2 = \frac{1}{\delta_1\delta_2(a_2\eta_2)^{\frac{\xi+1}{\delta_2}}}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!}\Gamma(\frac{\xi+1}{\delta_2}+\frac{\delta_1}{\delta_2}\nu)\omega^\nu \tag{36}$$

for $0 < a_1\eta_1 < (a_2\eta_2)^{\frac{\delta_1}{\delta_2}}$. Note that for $\delta_1 = \delta_2 = \delta$ the right side of (36) is a binomial series, giving a binomial sum for $a_1\eta_1 < a_2\eta_2$. The analytic continuation part is available from the poles of $\Gamma(\frac{\xi+1}{\delta_2} - \frac{s}{\delta_2})$. Replacing $\frac{s}{\delta_2}$ by s we have from (33), for $\hat{c} = [\delta_1(a_2\eta_2)^{\frac{\xi+1}{\delta_2}}]^{-1}$,

$$I_2 = \hat{c}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\frac{\delta_2}{\delta_1}s)\Gamma(\frac{\xi+1}{\delta_2}-s)[\frac{(a_1\eta_1)^{\frac{\delta_2}{\delta_1}}}{(a_2\eta_2)}]^{-s}\mathrm{d}s. \tag{37}$$

The poles of $\Gamma(\frac{\xi+1}{\delta_2} - s)$ are at $s = \frac{\xi+1}{\delta_2} + \nu, \nu = 0, 1, 2, ...$ Then

$$I_2 = \hat{c}\omega^{-\frac{\xi+1}{\delta_2}}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!}\Gamma(\frac{\xi+1}{\delta_1}+\frac{\delta_2}{\delta_1}\nu)\omega^{-\nu} \tag{38}$$

for $\omega > 1$. Thus, (36) and (38) give the series for all values of $\omega > 0$. Again, for $\delta_1 = \delta_2$, (38) reduces to a binomial sum.

3. One Factor with Negative Exponent

We can observe that in the pathway model for the cases of $q > 1$ and $q \to 1$, x and $\frac{1}{x}$ belong to the same family of functions. In other words both the situations x^δ and $x^{-\delta}$, with $\delta > 0$, are admissible cases. When $x^{-\delta}$ is there then the survival function will be of the form $1 - [1 + a(q-1)t^{-\delta}]^{-\frac{\eta}{q-1}+1}$. Let us again consider the case of two components, independently acting, or $k = 2$ where x_1 has a pathway model of beta type-2 and x_2 has an inverted type-2 beta pathway model. Then the h-th moment of the system survival time, written in terms of the survival function, is the following:

$$\begin{aligned} E(t^h) &= \int_0^\infty t^h f_x(t)\mathrm{d}t = h\int_0^\infty t^{h-1}S_x(t)\mathrm{d}t = h\int_0^\infty t^{h-1}[1+a_1(q_1-1)t^{\delta_1}]^{-\frac{\eta_1}{q_1-1}+1} \\ &\times \{1-[1+a_2(q_2-1)t^{-\delta_2}]^{-\frac{\eta_2}{q_2-1}+1}\}\mathrm{d}t. \end{aligned} \tag{39}$$

In order to evaluate the integral in (39) let us consider the general integral

$$g_2 = \int_0^\infty t^{\gamma-1}[1+a_1(q_1-1)t^{\delta_1}]^{-\frac{\eta_1}{q_1-1}+1}[1+a_2(q_2-1)t^{-\delta_2}]^{-\frac{\eta_2}{q_2-1}+1}\mathrm{d}t. \tag{40}$$

This can be evaluated with the help of Mellin convolution of a product. Let $x_1 > 0, x_2 > 0, u = x_1x_2, v = x_2$ or $x_1 = \frac{u}{v}$, Jacobian is $\frac{1}{v}$. Let the corresponding functions be $f_8(x_1)$ and $f_9(x_2)$. Then consider the integral

$$g_2 = \int_0^\infty \frac{1}{v}f_8(\frac{u}{v})f_9(v)\mathrm{d}v. \tag{41}$$

Then Mellin convolution of a product says that $M_{g_2}(s) = M_{f_8}(s)M_{f_9}(s)$ where s is the Mellin parameter. In terms of independently distributed real scalar positive random variables x_1 and x_2, with densities $f_8(x_1)$ and $f_9(x_2)$ respectively, g_2 will represent the density of the product $x_1x_2 = u$. Take $u = [a_2(q_2-1)]^{\frac{1}{\delta_2}}, x_2 = v$ and let

$$f_8(x_1) = [1+x_1^{\delta_2}]^{-\frac{\eta_2}{q_2-1}+1} \text{ and } f_9(x_2) = x_2^\gamma[1+a_1(q_1-1)x_2^{\delta_1}]^{-\frac{\eta_1}{q_1-1}+1}.$$

Then

$$f_8(\frac{u}{v}) = [1+a_2(q_2-1)v^{-\delta_2}]^{-\frac{\eta_2}{q_2-1}+1}$$

Then the Mellin transform of f_8, with Mellin parameter s, denoted by $M_{f_8}(s)$, is the following:

$$M_{f_8}(s) = \int_0^\infty x_1^{s-1} f_8(x_1)\mathrm{d}x_1 = \frac{\Gamma(\frac{s}{\delta_2})\Gamma(\frac{\eta_2}{q_2-1} - 1 - \frac{s}{\delta_2})}{\delta_2\Gamma(\frac{\eta_2}{q_2-1} - 1)}$$

for $\Re(s) > 0, \Re(\frac{\eta_2}{q_2-1} - 1 - \frac{s}{\delta_2}) > 0, \eta_2 + 1 - q_2 > 0$. But

$$\frac{1}{v} f_9(v) = v^{\gamma-1}[1 + a_1(q_1 - 1)v^{\delta_1}]^{-\frac{\eta_1}{q_1-1}+1}.$$

Then $\int_0^\infty \frac{1}{v} f_8(\frac{u}{v}) f_9(v)\mathrm{d}v$, with the above f_8 and f_9, agrees with the integral to be evaluated in (40). The Mellin transform of f_9 is given by

$$M_{f_9}(s) = \int_0^\infty x_2^{s-1} x_2^{\gamma}[1 + a_1(q_1 - 1)x_2^{\delta_1}]^{-\frac{\eta_1}{q_1-1}+1}\mathrm{d}x_2 = \frac{\Gamma(\frac{s+\gamma}{\delta_1})\Gamma(\frac{\eta_1}{q_1-1} - 1 - \frac{s+\gamma}{\delta_1})}{\delta_1[a_1(q_1 - 1)]^{\frac{\gamma+s}{\delta_1}}\Gamma(\frac{\eta_1}{q_1-1} - 1)}$$

for $\Re(s + \gamma) > 0, \Re(\frac{\eta_1}{q_1-1} - 1 - \frac{s+\gamma}{\delta_1}) > 0, \eta_1 > 0, \eta_1 + 1 - q_1 > 0$. Now, $M_{g_2}(s) = M_{f_8}(s)M_{f_9}(s)$. Then, taking the inverse Mellin transform, for $\tilde{c} = [\delta_1\delta_2[a_1(q_1 - 1)]^{\frac{\gamma}{\delta_1}}\Gamma(\frac{\eta_1}{q_1-1} - 1)\Gamma(\frac{\eta_2}{q_2-1} - 1)]^{-1}$,

$$\begin{aligned} g_2 &= \tilde{c}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(\frac{s}{\delta_2})\Gamma(\frac{s+\gamma}{\delta_1})\Gamma(\frac{\eta_2}{q_2-1} - 1 - \frac{s}{\delta_2}) \\ &\times \Gamma(\frac{\eta_1}{q_1-1} - 1 - \frac{\gamma+s}{\delta_1})\{[a_1(q_1 - 1)]^{\frac{1}{\delta_1}}[a_2(q_2 - 1)]^{\frac{1}{\delta_2}}\}^{-s}\mathrm{d}s \qquad (42) \\ &= \tilde{c}H_{2,2}^{2,2}\left[[a_1(q_1 - 1)]^{\frac{1}{\delta_1}}[a_2(q_2 - 1)]^{\frac{1}{\delta_2}}\Big|_{(0,\frac{1}{\delta_2}),(\frac{\gamma}{\delta_1},\frac{1}{\delta_1})}^{(2-\frac{\eta_2}{q_2-1},\frac{1}{\delta_2}),(2+\frac{\gamma}{\delta_1}-\frac{\eta_1}{q_1-1},\frac{1}{\delta_1})}\right] \qquad (43) \end{aligned}$$

for $[a_1(q_1 - 1)]^{\frac{1}{\delta_1}}[a_2(q_2 - 1)]^{\frac{1}{\delta_2}} < 1$. We can also obtain series forms here.

3.1. Limiting forms of this special case

When $q_1 \to 1$ and $q_2 \to 1$ then we have the following integral for the ρ-th moment of the time to failure:

$$I_3 = \rho\int_0^\infty t^{\rho-1}[\mathrm{e}^{-a_1\eta_1 t^{\delta_1}}][1 - \mathrm{e}^{-a_2\eta_2 t^{-\delta_2}}]\mathrm{d}t \qquad (44)$$

In order to evaluate (44) we will consider the following general integral:

$$g = \int_0^\infty t^{\gamma}\mathrm{e}^{-b_1 t^{\delta_1} - b_2 t^{-\delta_2}}\mathrm{d}t \qquad (45)$$

for $b_j > 0, \delta_j > 0, j = 1, 2$. This integral in (45) is connected to many problems in different fields. For $\delta_1 = 1, \delta_2 = 1$ it is the basic Krätzel integral, see [17], [18], [19] and [20]. For $\delta_1 = 1, \delta_2 = \frac{1}{2}$ it is the reaction-rate probability integral in nuclear reaction-rate theory, see [21]. The integrand in (45) for $\delta_1 = 1, \delta_2 = 1$, normalized, is the inverse Gaussian density in stochastic processes. Hence (45) is a generalization of all these basic integrals. This integral can be explicitly evaluated by treating it as a Mellin convolution of a product. Let $u = x_1x_2, v = x_2$ or $x_2 = v, x_1 = \frac{u}{v}$ with Jacobian $\frac{1}{v}$. Since the integrand in (45) is a product of positive integrable functions, by multiplying with appropriate normalizing constants, one can create statistical densities out of them. Hence we can treat the Mellin convolution of a product as the statistical problem of computing the density of a product of two statistically independently distributed real positive scalar random variables. Then $E(u^{s-1}) = [E(x_1^{s-1})][E(x_2^{s-1})]$ where E denotes the expected value, or in terms of the Mellin transforms, $M_g(s) = M_{f_{10}}(s)M_{f_{11}}(s)$ where s is the Mellin parameter. Then g has the structure

$$g = \int_v \frac{1}{v} f_{10}(\frac{u}{v}) f_{11}(v)\mathrm{d}v. \qquad (46)$$

Take

$$f_{10}(x_1) = \mathrm{e}^{-x_1^{\delta_2}} \Rightarrow f_{10}(\frac{u}{v}) = \mathrm{e}^{-b_2 v^{-\delta_2}} \tag{47}$$

where $u = b_2^{\frac{1}{\delta_2}} = (a_2\eta_2)^{\frac{1}{\delta_2}}$. Then the Mellin transform of f_1 is of the form

$$M_{f_{10}}(s) = \int_0^\infty \mathrm{e}^{-x^{\delta_2}}\mathrm{d}x = \frac{1}{\delta_2}\Gamma(\frac{s}{\delta_2}), \Re(s) > 0. \tag{48}$$

Take

$$\begin{aligned} f_{11}(x) &= x^{\gamma+1}\mathrm{e}^{-b_1 x^{\delta_1}} \Rightarrow & (49)\\ M_{f_{11}}(s) &= \int_0^\infty x^{\gamma+1+s-1}\mathrm{e}^{-b_1 x^{\delta_1}}\mathrm{d}x \\ &= [\delta_1 b_1^{\frac{s+\gamma+1}{\delta_1}}]^{-1}\Gamma(\frac{s+\gamma+1}{\delta_1}), \Re(s+\gamma+1) > 0. & (50)\end{aligned}$$

Observe that f_{10} from (46) and f_2 from (49), when substituted in (46) gives the integral to be evaluated in (50), which by the Mellin convolution of a product is the inverse Mellin transform of the product $M_{f_{10}}(s)M_{f_{11}}(s)$, available from (48) and (50). Therefore the integral in (50) is given by

$$\begin{aligned} g &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M_{f_{10}}(s)M_{f_{11}}(s)u^{-s}\mathrm{d}s \\ &= \bar{c}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(\frac{s}{\delta_2})\Gamma(\frac{\gamma+1}{\delta_1}+\frac{s}{\delta_1})(b_1^{\frac{1}{\delta_1}} b_2^{\frac{1}{\delta_2}})^{-s}\mathrm{d}s \\ &= \bar{c}H_{0,2}^{2,0}\left[b_1^{\frac{1}{\delta_1}} b_2^{\frac{1}{\delta_2}} \Big|_{(0,\frac{1}{\delta_2}),(\frac{\gamma+1}{\delta_1},\frac{1}{\delta_1})} \right], & (51)\end{aligned}$$

where $\bar{c} = [\delta_1\delta_2 b_1^{\frac{\gamma+1}{\delta_1}}]^{-1}$. When the poles are simple, (51) can be written as a sum of two series. When $\frac{1}{\delta_1} = m_1$ and $\frac{1}{\delta_2} = m_2$ where $m_1, m_2 = 1, 2, ...$ (positive integers) then the H−function in (51) can be written as a G−function and can be evaluated in explicit series forms. In the reaction rate probability integral $\delta_1 = 1$ and $\frac{1}{\delta_2} = 2$ and this problem is of the above type and explicit series forms may be seen from [21].

4. Multi-component stress-strength Reliability

A system containing more than one component is referred to as a multi-component system. It may consist of parallel or series components, or it may involve an intricate combination of both. Many real-world applications of MSS models may be found in areas including industrial processes, military technology, communication networks, etc. For example, a person may survive with only one healthy kidney, hence, kidney function in the human body is a one-out-of-two system. The MSS system functions when at least $s(1 \leq s \leq k)$ of its k identical and independent strength components function properly against a common strength. Let $X_1, X_2, \ldots, X_k$ be independent random variables with a common distribution function $F(.)$ and subjected to the common random stress Y with a distribution function $G(.)$. Thus the system reliability in a Multi-component stress strength model $R_{s,k}$ is given by

$$\begin{aligned} R_{s,k} &= P\left[\text{at least } s \text{ of} X_1, X_2, \ldots, X_k \text{ exceed } Y\right] \\ &= \sum_{i=s}^{k}\binom{k}{i}(P\left[X_i > Y\right])^i (P\left[X_i \leq Y\right])^{k-i} \\ &= \sum_{i=s}^{k}\binom{k}{i}\int_{-\infty}^{\infty}\left[1-F(y)\right]^i\left[F(y)\right]^{k-i}\mathrm{d}G(y) & (52)\end{aligned}$$

The multi−component system reliability given in equation (1) was first introduced by Bhattacharyya and Johnson [22]. After that, many authors have shown considerable interest in the multi-component stress−strength reliability for details refer [23], [24] etc.
In many complex systems that emerge in the domains of biology, chemistry, economics, geography, medicine, physics, etc., modelling and analysing lifetime data are crucial. The literature introduces a variety of q-type distributions for modeling lifetime data, the most prominent of which are the q-exponential, q-gamma, q-Gaussian etc, see [25] and [26], q-Weibull refer [27] and q-K-distribution, see [14]. The basic motivation for constructing statistical distributions for modelling lifetime data is the ability to model both monotonic and non-monotonic failure rates, even though the baseline failure rate may be monotonic. The Weibull distribution is most commonly used to describe lifetime data, which can only exhibit monotonic and constant shapes for its hazard rate function. However, the q-Weibull distribution can exhibit unimodal, bathtub-shaped, monotonically decreasing, monotonically increasing, and constant shapes for its hazard rate function. Hence, it is a useful generalization of the Weibull distribution. Here we discuss a classical inference on the multi-component stress-strength reliability when the stress and strength components are independent random variables distributed as (11). Then the Multi-component stress strength system reliability $R_{s,k}$ is given by

$$\begin{aligned} R_{s,k} &= \sum_{i=s}^{k}\binom{k}{i}\int_{-\infty}^{\infty}[1-F(y)]^{i}\,[F(y)]^{k-i}\,\mathrm{d}G(y) \\ &= \sum_{i=s}^{k}\binom{k}{i}\int_{0}^{\infty}\left[(1+\alpha(q-1)y^{\delta})^{-\frac{\eta_1}{q-1}+1}\right]^{i}\left[1-(1+\alpha(q-1)y^{\delta})^{-\frac{\eta_1}{q-1}+1}\right]^{k-i} \\ &\times\ \alpha\delta(\eta_2+1-q)y^{\delta-1}\left[(1+\alpha(q-1)y^{\delta})^{-\frac{\eta_2}{q-1}}\right]\mathrm{d}y \end{aligned} \tag{53}$$

After simplification, we get

$$R_{s,k}=\frac{(\eta_2+1-q)}{(\eta_1+1-q)}\sum_{i=s}^{k}\binom{k}{i}\mathbf{B}\Big(\frac{(\eta_2+1-q)}{(\eta_1+1-q)},k-i+1\Big)\ \text{for}\ \frac{\eta_2+1-q}{\eta_1+1-q}>0. \tag{54}$$

In this section, we created random samples from stress and strength variables for various parameter values and sample size combinations. In three scenarios, $(s,k)=(1,3),(2,6)$, and $(3,7)$ we estimated the MSS reliability. Tables 1 present the estimated values, bias, and mean square error (MSE).

multirow graphicx lscape

Table 1: *The MLE, Bias and SE of the estimator of $R_{s,k}$*

n	(s,k)=(1,3)			(s,k)=(2,6)			(s,k)=(3,7)		
	R-MLE	R-Bias	R-MSE	R-MLE	R-Bias	R-MSE	R-MLE	R-Bias	R-MSE
15	0.120119	0.020119	0.000405	0.064679	0.012048	0.000145	0.058502	0.013048	0.000170
20	0.093342	0.006658	0.000044	0.049380	0.003252	0.000011	0.037616	0.007838	0.000061
25	0.094298	0.005702	0.000033	0.049759	0.002873	0.000008	0.038526	0.006928	0.000048
30	0.104346	0.004346	0.000019	0.055473	0.002842	0.000008	0.050787	0.005333	0.000028
35	0.101434	0.001434	0.000002	0.053631	0.001000	0.000001	0.044765	0.000689	0.000001
n	(s,k)=(1,3)			(s,k)=(2,6)			(s,k)=(3,7)		
	R-MLE	R-Bias	R-MSE	R-MLE	R-Bias	R-MSE	R-MLE	R-Bias	R-MSE
50	0.193680	0.011861	0.000141	0.107388	0.007388	0.000055	0.093504	0.006548	0.000043
100	0.170321	0.011497	0.000132	0.093361	0.006639	0.000044	0.081143	0.005813	0.000034
125	0.186056	0.004238	0.000018	0.102703	0.002703	0.000007	0.089360	0.002404	0.000006
200	0.178269	0.003549	0.000013	0.097947	0.002054	0.000004	0.085158	0.001798	0.000003
250	0.181112	0.000706	0.000001	0.099623	0.000377	0.000000	0.086631	0.000325	0.000000

n	(s,k)=(1,3)			(s,k)=(2,6)			(s,k)=(3,7)		
	R-MLE	R-Bias	R-MSE	R-MLE	R-Bias	R-MSE	R-MLE	R-Bias	R-MSE
25	0.338753	0.018390	0.000338	0.205363	0.012029	0.000145	0.181581	0.010727	0.000115
150	0.347079	0.010064	0.000101	0.210166	0.007226	0.000052	0.185749	0.006559	0.000043
250	0.353725	0.003418	0.000012	0.215051	0.002341	0.000006	0.190202	0.002105	0.000004
300	0.356246	0.000897	0.000001	0.216886	0.000506	0.000000	0.191872	0.000435	0.000000
800	0.356669	0.000474	0.000000	0.217124	0.000268	0.000000	0.192077	0.000231	0.000000
n	(s,k)=(1,3)			(s,k)=(2,6)			(s,k)=(3,7)		
	R-MLE	R-Bias	R-MSE	R-MLE	R-Bias	R-MSE	R-MLE	R-Bias	R-MSE
25	0.233742	0.016258	0.000264	0.133105	0.009752	0.000095	0.116415	0.008585	0.000074
150	0.240789	0.009211	0.000085	0.136981	0.005876	0.000035	0.119772	0.005228	0.000027
250	0.253195	0.003195	0.000010	0.145040	0.002183	0.000005	0.126965	0.001965	0.000004
300	0.247085	0.002916	0.000009	0.140967	0.001891	0.000004	0.123313	0.001687	0.000003
800	0.248977	0.001023	0.000001	0.142251	0.000606	0.000000	0.124468	0.000532	0.000000
n	(s,k)=(1,3)			(s,k)=(2,6)			(s,k)=(3,7)		
	R-MLE	R-Bias	R-MSE	R-MLE	R-Bias	R-MSE	R-MLE	R-Bias	R-MSE
50	0.534931	0.034931	0.001220	0.367358	0.034024	0.001158	0.332781	0.032781	0.001075
100	0.478824	0.021176	0.000448	0.315918	0.017415	0.000303	0.283795	0.016205	0.000263
200	0.488284	0.011716	0.000137	0.323447	0.009887	0.000098	0.290755	0.009245	0.000086
450	0.496283	0.003717	0.000014	0.330132	0.003202	0.000010	0.296994	0.003006	0.000009
700	0.499025	0.000975	0.000001	0.332562	0.000771	0.000001	0.299289	0.000711	0.000001
n	(s,k)=(1,3)			(s,k)=(2,6)			(s,k)=(3,7)		
	R-MLE	R-Bias	R-MSE	R-MLE	R-Bias	R-MSE	R-MLE	R-Bias	R-MSE
50	0.357590	0.024256	0.000588	0.218419	0.018419	0.000339	0.193363	0.016892	0.000285
100	0.315656	0.017677	0.000313	0.188037	0.011963	0.000143	0.165723	0.010748	0.000116
200	0.326700	0.006633	0.000044	0.195442	0.004558	0.000021	0.172365	0.004106	0.000017
450	0.328412	0.004921	0.000024	0.196541	0.003459	0.000012	0.173343	0.003128	0.000010
700	0.332360	0.000973	0.000001	0.199353	0.000647	0.000000	0.175891	0.000580	0.000000
n	(s,k)=(1,3)			(s,k)=(2,6)			(s,k)=(3,7)		
	R-MLE	R-Bias	R-MSE	R-MLE	R-Bias	R-MSE	R-MLE	R-Bias	R-MSE
50	0.177839	0.011172	0.000125	0.097755	0.006846	0.000047	0.084999	0.006052	0.000037
100	0.155884	0.010783	0.000116	0.084773	0.006136	0.000038	0.073587	0.005361	0.000029
300	0.170740	0.004074	0.000017	0.093368	0.002459	0.000006	0.081116	0.002169	0.000005
500	0.162740	0.003926	0.000015	0.088604	0.002305	0.000005	0.076923	0.002024	0.000004
700	0.166253	0.000414	0.000000	0.090677	0.000232	0.000000	0.078745	0.000202	0.000000

5. Real Data Application

In this section, we explore an actual data set to illustrate the flexibility of the proposed model. The information displays, in months, how long 128 bladder cancer patients were in remission. The data set is given in Table 2.

Table 2: *Remission times of bladder cancer patients data*

0.08	2.09	13.29	0.4	2.26	3.57	5.06	7.09	9.22	13.8	25.74	0.5
3.48	4.87	23.63	0.2	2.23	6.94	8.66	13.11	3.52	4.98	6.97	9.02
3.88	5.32	7.39	10.34	14.83	34.26	0.9	2.69	4.18	5.34	7.59	10.66
2.46	3.64	5.09	7.26	9.47	14.24	25.82	0.51	2.54	3.7	5.17	7.28
15.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01	1.19	2.75
9.74	14.76	26.31	0.81	2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64
11.79	18.1	1.46	4.4	5.85	8.26	11.98	19.13	1.76	3.25	4.5	6.25
79.05	1.35	6.76	17.14	2.87	5.62	7.87	11.64	17.36	1.4	3.02	4.34
5.71	7.93	22.69	4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33	5.49
7.66	11.25	21.73	2.07	3.36	6.93	8.37	12.02	2.02	12.07	20.28	2.02
3.36	12.03	3.31	4.51	6.54	8.53	8.65	12.63				

We compare the proposed model's goodness of fit to a few competing models, such as Weibull, Frechet Weibull, transmuted Weibull, and modified Weibull (MW), using a few discrimination criteria, such as the Akaike Information Criterion (AIC), Anderson Darling test (AD-test), Cram©r-von Mises test (CRVM), and Kolmogorov-Smirnov test with its *p*-value. The MLEs of the parameters, as well as the values of the AIC, are provided in Tables 3 and 4, respectively. These findings suggest that the proposed model is the best model because it has the lowest test statistic values among all fitted models. The plots of the fitted PDF, CDF, P-P plot, and Q-Q plot for the proposed distribution and Weibull distribution are displayed in Figure 9.

Table 3: *The estimated value of the parameters of the fitted model.*

Generalized q-Weibull	q=2.5925	$\eta = 4.8920$	$\alpha = 0.0179$	$\delta = 1.4273$
Weibull	$\alpha = 1.0478$	$\beta = 9.5607$	—-	—
Frechet Weibull (FW	$\alpha = 1.1446$	$\beta = 1.881$	—	—
Transmuted Weibull	$\alpha = 1.1333$	$\beta = 14.6198$	$\lambda = 0.7449$	—
Modified Weibull	alpha=1.3172	$\beta = 0 : 0938$	$\lambda = 1.4783$	—

Table 4: *The value of AIC, AD-test, CRVM-test, KS-test, and p-value of the fitted model.*

Model	AIC	AD-test	CRVM-test	KS-test	p=value
Generalized q-Weibull	827.4798	0.12177	0.01758	0.03504	0.99943
Weibull	832.174	0.957709	0.153703	0.0700169	0.556965
Frechet Weibull (FW)	896.002	6.11825	0.978722	0.140799	0.0125018
Transmuted Weibull	829.917	0.560038	0.0879162	0.0587652	0.76866
Modified Weibull	834.174	0.957709	0.153703	0.0700169	0.556965

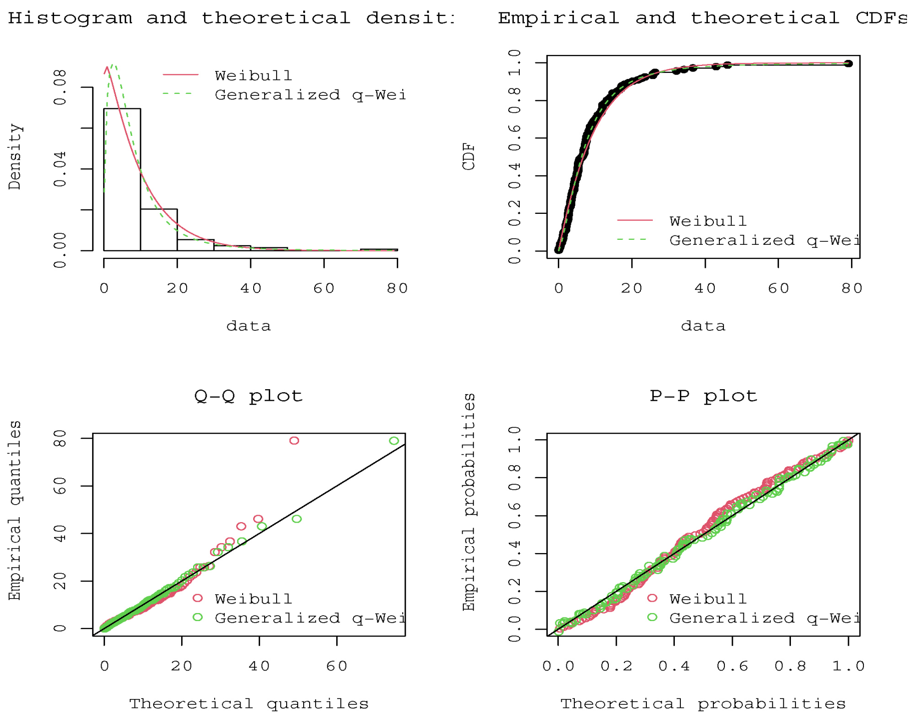

Figure 9: The histogram and theoretical densities, empirical and theoretical CDFs, Q-Q plots, P-P plots of the fitted data.

6. Conclusion

In this study, we take into account a system with k-connected components in series. The lifetimes of the components, $X_1, ..., X_k$, are randomly distributed and have pathway densities for the pathway parameters $q < 1$, $q > 1$, or $q \to 1$. Then, the survival function, hazard function, expected time to failure, and general moments of $x = \min\{X_1, X_2, \ldots, X_n\}$ are computed. It is demonstrated that the hazard function can take on various shapes, including a bathtub shape. The estimation of stress-strength reliability is assessed through the maximum likelihood estimation technique when both stress and strength variables conform to the pathway model. Remission time data from cancer patients is examined to see how the model is relevant in practical situations. The proposed distribution consistently provides better fits for real data compared to other models.

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