CONFIDENCE INTERVALS FOR THE PARAMETER OF THE IWUEZE DISTRIBUTION WITH APPLICATIONS TO MEDICAL AND ENGINEERING DATA

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Abstract

One of the lifetime distributions is the Iwueze distribution, which is constructed by combining the exponential and gamma distributions. In this paper, confidence intervals (CIs) are proposed for the parameter of the Iwueze distribution using the likelihood-based, Wald-type, bootstrap-t, and bias-corrected and accelerated (BCa) bootstrap methods. We evaluated the performance of the proposed CI methods through Monte Carlo simulation in terms of their coverage probability (CP) and average length (AL) in various scenarios. Furthermore, we had also derived the explicit formula for the Wald-type CI, which is straightforward for computation. The simulation results showed that the likelihood-based and Wald-type CIs returned satisfactory results according to coverage probabilities, even for the setting of small sample sizes. On the other hand, both the bootstrap-t and BCa bootstrap CIs yield CPs lower than the nominal confidence level when sample sizes are small. However, as the sample sizes increase, the CP of all CIs tend to approach the nominal confidence level. The parameter values also have a minor influence on the CP of all CIs when the sample size is fixed. Moreover, the AL of all CIs decreases as the sample size increases. The Wald-type and likelihood-based CIs have very similar ALs for all parameter values. In general, the bootstrap-t CI tends to yield the shortest interval. The effectiveness of all CIs was demonstrated by applying them to medical and engineering data, yielding results consistent with those of the simulation study.

Keywords: lifetime distribution, interval estimation, likelihood, Wald, bootstrap

I. Introduction

In reliability and lifetime data analysis, lifetime distributions are statistical distributions that can be used to describe the behavioral structure of lifetime data. Lifetime distributions are utilized to represent the duration before the occurrence of a significant event, such as failure or incidence [1]. The field of lifetime data analysis has had substantial growth and progress in terms of technique, theory, and application. The distribution theory focuses on the capacity to easily handle and adapt to modeling lifespan data. While a tractable probability distribution could be useful for replicating random samples, its practical value to businesses lies in its flexibility [2]. This suggests that while tractable distributions are desirable, more complex ones must be created to support relevant applications.

Many lifetime distributions have been proposed in statistics in the past few decades. Nevertheless, these distributions frequently do not offer a precise match because of either their basic distributional properties or the structure of the lifetime data. Several distribution theory experts are trying to suggest a new lifetime distribution consistent with the stochastic nature of lifespan data. Before 1958, the exponential distribution was the only lifetime distribution accessible for the analysis and modeling of lifetime data. The Lindley distribution was presented by Lindley [3] as an alternative lifespan distribution. Based on their comprehensive analysis of the statistical properties and practical uses, Ghitanty et al. [4] determined that the Lindley distribution offers a much superior match compared to the exponential distribution. Shanker et al. [5] observed that when analyzing exponential and Lindley distributions, there is a significant competition between these two distributions. However, they also identified specific datasets in which neither distribution provided a sufficient fit. Shanker [6, 7] proposed two new one-parameter lifespan distributions, named Shanker distribution and Akash distribution. These distributions demonstrated better fit to data than both exponential and Lindley distributions. Furthermore, the Lindley, exponential, Shanker, and Akash distributions were thoroughly examined by Shanker and Fesshaye [8]. They discovered that while these distributions work well for most datasets, there are some that still do not provide the best fit. In addition, Shanker [9] introduced the Sujatha distribution, which has a considerably better fit when compared to the exponential, Lindley, Shanker, and Akash distributions. Shanker [10] proposed the Garima distribution, a single-parameter lifespan distribution, as a suitable statistical model for data collected from the behavioral sciences. However, this distribution likewise fails to provide a satisfactory match for several actual lifespan datasets.

The current paper is to identify a distribution that can accurately depict the diversity within the data sets while remaining flexible and tractable. When a distribution does not provide a sufficient match, many researchers choose to transform the dataset to meet the assumptions of the distribution. Nevertheless, this approach is unsuitable as it leads to the loss of the dataset's inherent characteristics. Some researchers prefer to adjust the distribution by incorporating extra shape or scale parameters to better fit with the characteristics of the data set. However, in cases where the current distributions are unable to generate a suitable fit, it is more advantageous to seek out an alternative distribution that can. This approach involves refraining from transforming the original dataset or modifying the distribution to fit the dataset. Recently, Elechi et al. [11] proposed the Iwueze distribution, a five-component mixture of exponential and gamma distributions with a constant scale parameter, and different shape parameters 2, 3, 4, and 5. This distribution has superior efficiency in comparison to other one-parameter distributions. The flexibility of the Iwueze distribution is demonstrated through its application to relief times of patients receiving an analgesic.

In the review literature, there is no research study for estimating the confidence intervals (CIs) for the parameter of the Iwueze distribution. Therefore, the objective of the paper is to propose the CIs for the parameter of the Iwueze distribution in four methods, namely, likelihood-based CI, Wald-type CI, bootstrap-t interval, and bias-corrected and accelerated (BCa) bootstrap CI. We conduct a simulation study and analyze real data sets to compare the performance of CIs for the parameter of the Iwueze distribution.

The following is the outline of the paper. In Section 2, the Iwueze distribution are explained. Section 3 involves the computation of the likelihood-based, Wald-type, bootstrap-t, and BCa bootstrap CIs for the parameter of the Iwueze distribution. Section 4 evaluates the effectiveness of the proposed CIs by utilizing Monte Carlo simulation in various circumstances. Section 5 contains two numerical examples. Ultimately, the final section of the paper contains the discussion and conclusions.

II. The Iwueze Distribution

The Iwueze distribution is obtained by combining the exponential and gamma distributions using appropriate mixing probabilities. The gamma distribution has a fixed scale parameter θ and four different shape parameters: 2, 3, 4, and 5. Let X be a random variable which follow the Iwueze distribution with parameter θ. The probability density function (pdf) of the Iwueze distribution can be obtained by utilizing a mixture model with five component mixing probabilities. The pdf is given by

$$f(x;\theta)=\frac{\theta^5}{\theta^4+2\theta^3+6\theta^2+12\theta+24}(1+x+x^2)^2\,e^{-\theta x},\quad x>0,\theta>0.$$

Figure 1 shows the plots of the Iwueze distribution pdf with several parameter values θ. The mean (or the first central moment) and variance (or the second central moment) of X are given by

$$E(X)=\mu=\frac{\theta^4+2\left[2\theta^3+3\left(3\theta^2+4(2\theta+5)\right)\right]}{\theta\left(\theta^4+2\theta^3+6\theta^2+12\theta+24\right)},$$

and

$$Var(X)=\sigma^2=\frac{\left(\theta^8+8\theta^7+56\theta^6+240\theta^5+876\theta^4+1344\theta^3+2304\theta^2+2880\theta+2880\right)}{\theta^2\left(\theta^4+2\theta^3+6\theta^2+12\theta+24\right)^2}.$$

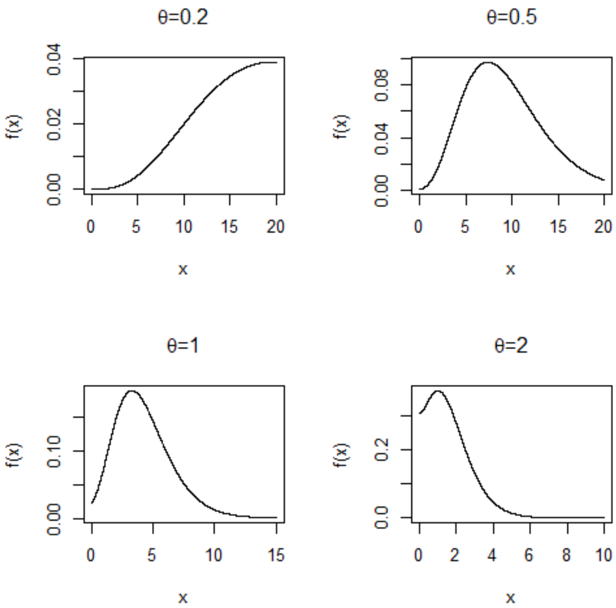

Figure 1: *Plots of the pdf of the Iwueze distribution for θ = 0.2, 0.5, 1, and 2*

The log-likelihood function $\log L(\theta\,|\,x_i)$, is maximized to obtain the point estimator of θ. Therefore, the maximum likelihood (ML) estimator for θ of the Iwueze distribution is derived by the following processes:

$$\frac{\partial}{\partial\theta}\log L(\theta\,|\,x_i)=\frac{\partial}{\partial\theta}\left[5n\log(\theta)-n\log(\theta^4+2\theta^3+6\theta^2+12\theta+24)+\sum_{i=1}^{n}\log\left[1+x_i+x_i^2\right]^2-\theta\sum_{i=1}^{n}x_i\right]$$

$$=\frac{5n}{\theta}-\frac{n\left(4\theta^3+6\theta^2+12\theta+12\right)}{\theta^4+2\theta^3+6\theta^2+12\theta+24}-\sum_{i=1}^{n}x_i.$$

The subsequent equation is a nonlinear equation obtained through the process of solving the equation $\frac{\partial}{\partial\theta}\log L(x_i;\theta)\overset{\text{set}}{=}0$ for θ,

$$\frac{5n}{\theta}-\frac{n\left(4\theta^3+6\theta^2+12\theta+12\right)}{\theta^4+2\theta^3+6\theta^2+12\theta+24}-\sum_{i=1}^{n}x_i\overset{\text{set}}{=}0.$$

Due to the absence of a closed-form solution for the ML estimator of parameter θ, numerical iteration methods are employed to solve the associated non-linear equation [12]. In this study, the maxLik package [13] was utilized to perform ML estimation using the Newton-Raphson technique in the RStudio program [14].

III. Confidence Intervals for the Parameter of the Iwueze Distribution

I. Likelihood-based Confidence Interval

The likelihood function for the Iwueze distribution, $L(\theta\,|\,x)$, is a function of the parameter θ, given the observed data x. It encapsulates the probability of observing the given data under various hypothetical values of θ. After solving $\frac{\partial}{\partial\theta}\log L(\theta\,|x)\overset{\text{set}}{=}0$, the ML estimator of θ, $\hat{\theta}_{ML}$, will be obtained, and this is the most "likely" estimate given the observed data.

The likelihood-based CI is then constructed around this ML estimator. The process begins by defining a likelihood ratio $\lambda(\theta)$ as $\lambda(\theta)=L(\theta\,|\,x)\big/L(\hat{\theta}\,|\,x)$. Under regular conditions, as per the Wilks' theorem, $-2\log\lambda(\theta)$ follows approximately a chi-square distribution with degrees of freedom equal to the number of parameters being estimated. Therefore, the CI for θ at $(1-\alpha)100\%$ confidence level is given by

$$\left\{\theta\left|-2\log\frac{L(\theta\,|\,x)}{L(\hat{\theta}\,|\,x)}\le\chi^2_{1-\alpha,1}\right.\right\}=\left\{\theta\left|-2\log\left[\frac{\theta^{5n}\left(\hat{\theta}^4+2\hat{\theta}^3+6\hat{\theta}^2+12\hat{\theta}+24\right)^n}{\hat{\theta}^{5n}\left(\theta^4+2\theta^3+6\theta^2+12\theta+24\right)^n}\exp\left(-\theta\sum_{i=1}^{n}x_i+\hat{\theta}\sum_{i=1}^{n}x_i\right)\right]\le\chi^2_{1-\alpha,1}\right.\right\},$$

where $\chi^2_{1-\alpha,1}$ is the critical value from the chi-square distribution with 1 degree of freedom [15,16]. In the specific case of the Iwueze distribution, the likelihood ratio test becomes more intricate due to the composite nature of the distribution. The gamma component, characterized by a scale parameter and shape parameters, adds layers of complexity to the likelihood function, necessitating advanced computational techniques, like numerical optimization, for effective ML estimator calculation and CI construction.

Brent's method, a root-finding algorithm often used in optimization, is used for finding the maximum MLE in the Iwueze distribution. It is an advanced technique that combines the bisection method, the secant method, and inverse quadratic interpolation [17]. Given that

$$f(\theta)=\frac{\partial}{\partial\theta}\log L(\theta\,|\,x)=\frac{5n}{\theta}-\frac{n\left(4\theta^3+6\theta^2+12\theta+12\right)}{\theta^4+2\theta^3+6\theta^2+12\theta+24}-\sum_{i=1}^{n}x_i\overset{\text{set}}{=}0,$$

Brent's method seeks θ such that $f(\theta)=0$. The method combines bracketing methods and open methods. Initially, if $f(a)f(b)<0$ it starts with the bisection method to ensure reliability. Then, depending on the function's behavior, it switches between the secant method (linear interpolation):

$$\theta_{\text{second}}=\theta_n-f(\theta_n)\frac{\theta_n-\theta_{n-1}}{f(\theta_n)-f(\theta_{n-1})},$$

and inverse quadratic interpolation (quadratic polynomial interpolation):

$$\theta_{\text{quad}}=\frac{f(\theta_{n-1})f(\theta_{n-2})}{\left(f(\theta_n)-f(\theta_{n-1})\right)\left(f(\theta_n)-f(\theta_{n-2})\right)}\theta_n+\ldots$$

The method iteratively refines the estimate of the root, switching methods based on which provides a more accurate or stable estimate [18,19]. Figure 2 shows the plot $-2\log\lambda(\theta)$ versus θ (solid blue line), $\chi^2_{0.95,1}$ (dashed red line), and 95% likelihood-based CI (solid green line) when a random sample of size 20 sampled from the Iwueze distribution with $\theta = 1$.

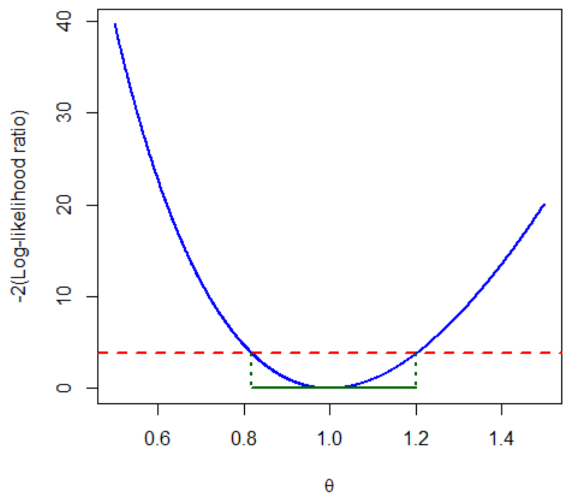

Figure 2: *The plot of* $-2\log\lambda(\theta)$ *versus* θ

Because the cut-point for constructing a likelihood-based CI often involves the use of an asymptotic distribution like the chi-square distribution, this reliance is grounded in Wilks theorem, the effectiveness of the likelihood-based CI in approximating the true parameter values does rely on the assumption that the sample size is sufficiently large for the asymptotic approximation to be valid. However, likelihood-based CI does not always rely on large sample sizes. It can provide accurate interval estimates even in cases with smaller sample sizes, assuming the likelihood function behaves well.

II. Wald-type Confidence Interval

The Wald-type CI is a fundamental statistical tool used for estimating the uncertainty associated with a parameter estimate in a probability distribution. Central to this method is the ML estimate of the parameter, denoted as $\hat{\theta}$ for the Iwueze distribution. The foundation of the Wald-type CI lies in the quadratic approximation of the log-likelihood function, $L(\theta \mid x)$, which can be expanded using a Taylor series around $\hat{\theta}$. The Wald statistic approximates the log-likelihood ratio when expanded to the second-order term around the ML estimate, with the first-order term equal to zero at the ML estimate as follows:

$$\log L(\theta \mid x) \approx \log L(\hat{\theta} \mid x) + \left(\theta - \hat{\theta}\right)\frac{\partial}{\partial\theta}\log L(\theta \mid x)\Big|_{\theta=\hat{\theta}} + \frac{1}{2}\left(\theta - \hat{\theta}\right)^2 \frac{\partial^2}{\partial\theta^2}\log L(\theta \mid x)\Big|_{\theta=\hat{\theta}}$$

$$\log\frac{L(\theta \mid x)}{L(\hat{\theta} \mid x)} \approx \frac{1}{2}\left(\theta - \hat{\theta}\right)^2 \frac{\partial^2}{\partial\theta^2}\log L(\theta \mid x)\Big|_{\theta=\hat{\theta}}$$

$$-2\log\frac{L(\theta \mid x)}{L(\hat{\theta} \mid x)} \approx \left(\theta - \hat{\theta}\right)^2 I\left(\hat{\theta}\right),$$

where $I(\hat{\theta})$ si the estimated observed Fisher information. The Wald statistic can thus serve as an approximation to the LRT statistic, particularly when the sample size is large enough for the asymptotic properties to hold, leading to a quadratic approximation of the log-likelihood ratio [20-22].

For the Iwueze distribution, the observed Fisher information is as follows:

$$\frac{\partial}{\partial\theta}\log L(\theta \mid x) = \frac{5n}{\theta} - \frac{n\left(4\theta^3+6\theta^2+12\theta+12\right)}{\theta^4+2\theta^3+6\theta^2+12\theta+24} - \sum_{i=1}^{n} x_i,$$

$$\frac{\partial^2}{\partial\theta^2}\log L(\theta \mid x) = -\frac{5n}{\theta^2} - \frac{12n(\theta^2+\theta+1)}{\theta^4+2\theta^3+6\theta^2+12\theta+24} + \frac{n(4\theta^3+6\theta^2+12\theta+12)^2}{(\theta^4+2\theta^3+6\theta^2+12\theta+24)^2}.$$

Thus, the estimated Fisher information is as follows:

$$I(\hat{\theta}) = \frac{5n}{\hat{\theta}^2} + \frac{12n(\hat{\theta}^2+\hat{\theta}+1)}{\hat{\theta}^4+2\hat{\theta}^3+6\hat{\theta}^2+12\hat{\theta}+24} - \frac{n(4\hat{\theta}^3+6\hat{\theta}^2+12\hat{\theta}+12)^2}{(\hat{\theta}^4+2\hat{\theta}^3+6\hat{\theta}^2+12\hat{\theta}+24)^2},$$

and the Wald-type CI for θ at $(1-\alpha)100\%$ confidence level is given by

$$\hat{\theta} \pm z_{1-\frac{\alpha}{2}}\sqrt{I^{-1}(\hat{\theta})},$$

where $z_{1-(\alpha/2)}$ denotes the $\left(1-(\alpha/2)\right)^{\text{th}}$ quantile of the standard normal distribution.

III. Bootstrap-t Confidence Interval

The bootstrap-t CI emerges as an advanced technique designed to calibrate the CI for an estimated parameter by incorporating the inherent variability of the estimate's standard error. This method extends the bootstrap percentile method by factoring in the fluctuation of the standard error, thereby enhancing the accuracy and reliability of the interval, particularly in small sample contexts or when dealing with estimators that deviate from normality [23-25]. The algorithmic foundation of the bootstrap-t CI can be delineated in the following steps:

1) Initialization: Commence with a sample $X_1,\ldots,X_n$ from which the parameter estimate $\hat{\theta}$ and its standard error $S.E.(\hat{\theta})$.

2) Bootstrap Resampling: Generate $B=1000$ bootstrap samples, $X_1^*,\ldots,X_n^*$, by random sampling with replacement from the original dataset.

3) Statistical Computation: For each bootstrap sample, calculate the bootstrap replicate of the estimator, denoted as $\hat{\theta}^*$, and its associated standard error $S.E.(\hat{\theta}^*)$.

4) Studentization: Construct the bootstrap-t statistic for each replicate as

$$t^*(X,\hat{\theta},\hat{\theta}^*) = \frac{\hat{\theta}^*-\hat{\theta}}{\sqrt{I^{-1}(\hat{\theta}^*)}}.$$

This studentized statistic adjusts for the variability in the standard error of the bootstrap estimate.

5) Repeating this process $B=1000$ times yields an empirical distribution of the estimator; from which we can estimate the distribution of the pivotal quantity.

6) Empirical Distribution: Formulate the empirical distribution of the bootstrap-t statistics from the ensemble of B replicates.

7) Quantile Extraction: Ascertain the critical values, $t^*_{(\alpha/2)}$ and $t^*_{(1-(\alpha/2))}$, which correspond to the $\alpha/2$ and $1-(\alpha/2)$ quantiles of the empirical bootstrap-t distribution,

$$\frac{\#\left(t^*(X,\hat{\theta},\hat{\theta}^*)\le t^*_{(\alpha/2)}\right)}{B} = \alpha \text{ and } \frac{\#\left(t^*(X,\hat{\theta},\hat{\theta}^*)\le t^*_{(1-(\alpha/2))}\right)}{B} = 1-(\alpha/2).$$

8) Interval Construction: The bootstrap-t CI is then articulated as:

$$\left[\hat{\theta}+t^*_{(\alpha/2)}\sqrt{I^{-1}(\hat{\theta})},\hat{\theta}+t^*_{(1-(\alpha/2))}\sqrt{I^{-1}(\hat{\theta})}\right].$$

IV. Bias-Corrected and Accelerated (BCa) Bootstrap Confidence Interval

The BCa bootstrap CI is a technique used for constructing CIs. This method refines the basic bootstrap procedure by introducing adjustments for both bias and skewness in the distribution of bootstrap estimates. Bias is calculated based on the proportion of bootstrap estimates that are less than the observed estimate, and this information is then used to adjust the percentiles of the CI. An acceleration parameter is incorporated to account for the skewness or asymmetry of the bootstrap distribution [26-28]. The algorithm is as follows:

1) Bootstrap Resampling: Draw $B=$ 1000 bootstrap samples from the empirical distribution of the original sample and calculate the bootstrap estimates $\hat{\theta}_b^*$, for $b=1,2,...,1000.$

2) Bias Correction (z_0): Determine the proportion of bootstrap estimates that are less than the original estimate $\hat{\theta}$, denoted p. The bias correction factor z_0 is the quantile of the standard normal distribution corresponding to p.

3) Acceleration (a): Calculate the acceleration value a which accounts for the asymmetry of the estimator's distribution. This is often estimated by the jackknife or other methods that quantify the skewness of the sampling distribution.

4) Adjusted Percentiles: Transform the bias-corrected normal deviates to adjust the percentiles for constructing the CI. The adjusted percentiles are given by

$$p_L^* = \Phi\left(z_0 + \frac{z_0 + z_{\alpha/2}}{1 - a\left(z_0 + z_{\alpha/2}\right)} \right)$$

and

$$p_U^* = \Phi\left(z_0 + \frac{z_0 + z_{1-(\alpha/2)}}{1 - a\left(z_0 + z_{1-(\alpha/2)}\right)} \right),$$

where Φ is the standard normal cumulative distribution function, and $z_{\alpha/2}$ and $z_{1-\alpha/2}$ are the $(\alpha/2)$-th and $\left(1-(\alpha/2)\right)$-th quantiles of the standard normal distribution, respectively.

5) CI Construction: The BCa bootstrap CI is constructed using the percentiles p_L^* and p_U^* to extract the corresponding quantiles from the bootstrap distribution of $\hat{\theta}_{ML}^*$. The formula is as follows

$$\left[\hat{\theta}^*_{\left(P_L^*\right)}, \hat{\theta}^*_{\left(P_U^*\right)} \right],$$

where $\hat{\theta}^*_{(P_L^*)}$ and $\hat{\theta}^*_{\left(P_U^*\right)}$ are $(p_L^*)^{\text{th}}$ and $(p_U^*)^{\text{th}}$ quantiles of the bootstrap estimates $\hat{\theta}_b^*$.

IV. Simulation Study and Results

This simulation study evaluates the effectiveness of 95% confidence interval (CI) construction methods in different scenarios. Our focus includes sample sizes, parameter values, coverage probability (CP), and the average length (AL) of the intervals. We vary sample sizes (n) at 10, 20, 30, 40, 100, 200, and 500, while also altering the distribution's parameter values (θ) to 0.2, 0.5, 0.75, 1, 1.5, 2, and 3. The CPs and ALs of the CIs are estimated using Monte Carlo simulations with 2,000 replications.

Table 1. *Coverage probability and average length of the 95% CIs for the parameter of the Iwueze distribution*

n	θ	Coverage probability				Average length			
		Likelihood	Wald	Bootstrap-t	BCa	Likelihood	Wald	Bootstrap-t	BCa
10	0.2	0.953	0.950	0.902	0.905	0.1135	0.1132	0.1014	0.1065
	0.3	0.956	0.952	0.899	0.902	0.1691	0.1688	0.1512	0.1582
	0.5	0.950	0.950	0.896	0.893	0.2811	0.2806	0.2536	0.2661
	0.75	0.949	0.951	0.898	0.903	0.4177	0.4170	0.3770	0.3940
	1	0.952	0.951	0.905	0.903	0.5525	0.5515	0.5007	0.5237
	1.5	0.951	0.957	0.906	0.905	0.8285	0.8252	0.7483	0.7934
	2	0.953	0.959	0.897	0.897	1.1382	1.1303	1.0045	1.0872
	2.5	0.958	0.961	0.908	0.908	1.4943	1.4856	1.3334	1.4720
20	0.2	0.946	0.950	0.923	0.926	0.0792	0.0791	0.0749	0.0763
	0.3	0.961	0.960	0.931	0.927	0.1185	0.1184	0.1120	0.1142
	0.5	0.949	0.945	0.920	0.921	0.1972	0.1971	0.1881	0.1922
	0.75	0.942	0.944	0.919	0.917	0.2931	0.2929	0.2780	0.2833
	1	0.943	0.946	0.925	0.921	0.3872	0.3868	0.3674	0.3740
	1.5	0.945	0.944	0.927	0.926	0.5768	0.5757	0.5458	0.5603
	2	0.947	0.945	0.921	0.923	0.7888	0.7862	0.7495	0.7749
	2.5	0.955	0.964	0.930	0.933	1.0421	1.0372	0.9785	1.0177
30	0.2	0.946	0.948	0.936	0.934	0.0644	0.0643	0.0622	0.0631
	0.3	0.959	0.957	0.940	0.939	0.0967	0.0966	0.0933	0.0947
	0.5	0.945	0.949	0.932	0.935	0.1601	0.1600	0.1542	0.1564
	0.75	0.951	0.952	0.936	0.937	0.2385	0.2383	0.2309	0.2335
	1	0.957	0.957	0.939	0.939	0.3153	0.3151	0.3036	0.3074
	1.5	0.955	0.955	0.938	0.940	0.4712	0.4706	0.4568	0.4648
	2	0.959	0.958	0.940	0.942	0.6419	0.6405	0.6153	0.6281
	2.5	0.948	0.952	0.934	0.939	0.8384	0.8358	0.8043	0.8264
50	0.2	0.957	0.958	0.946	0.946	0.0498	0.0497	0.0485	0.0490
	0.3	0.952	0.952	0.942	0.940	0.0746	0.0746	0.0728	0.0735
	0.5	0.961	0.958	0.945	0.950	0.1234	0.1234	0.1204	0.1216
	0.75	0.951	0.953	0.935	0.936	0.1839	0.1838	0.1800	0.1819
	1	0.953	0.952	0.940	0.938	0.2434	0.2433	0.2370	0.2393
	1.5	0.947	0.949	0.939	0.941	0.3640	0.3637	0.3557	0.3596
	2	0.955	0.958	0.950	0.950	0.4930	0.4923	0.4830	0.4906
	2.5	0.950	0.952	0.939	0.940	0.6448	0.6436	0.6271	0.6392
100	0.2	0.949	0.947	0.943	0.943	0.0351	0.0350	0.0346	0.0349
	0.3	0.948	0.950	0.938	0.945	0.0526	0.0526	0.0520	0.0525
	0.5	0.946	0.948	0.941	0.944	0.0874	0.0874	0.0860	0.0868
	0.75	0.945	0.946	0.939	0.939	0.1299	0.1298	0.1281	0.1291
	1	0.950	0.951	0.949	0.949	0.1715	0.1715	0.1687	0.1701
	1.5	0.944	0.945	0.938	0.939	0.2562	0.2561	0.2523	0.2549
	2	0.946	0.946	0.943	0.944	0.3481	0.3479	0.3443	0.3476
	2.5	0.948	0.948	0.941	0.947	0.4560	0.4556	0.4492	0.4551
200	0.2	0.951	0.949	0.948	0.948	0.0248	0.0248	0.0246	0.0248
	0.3	0.954	0.952	0.945	0.946	0.0370	0.0370	0.0366	0.0369

n	θ	Coverage probability				Average length			
		Likelihood	Wald	Bootstrap-t	BCa	Likelihood	Wald	Bootstrap-t	BCa
	0.5	0.951	0.949	0.947	0.947	0.0616	0.0616	0.0612	0.0617
	0.75	0.949	0.949	0.948	0.950	0.0917	0.0917	0.0912	0.0920
	1	0.944	0.948	0.940	0.943	0.1215	0.1215	0.1203	0.1212
	1.5	0.951	0.951	0.944	0.948	0.1809	0.1808	0.1793	0.1809
	2	0.956	0.957	0.951	0.951	0.2461	0.2460	0.2438	0.2460
	2.5	0.942	0.944	0.936	0.935	0.3209	0.3207	0.3184	0.3213
500	0.2	0.945	0.944	0.940	0.943	0.0156	0.0156	0.0156	0.0157
	0.3	0.948	0.949	0.944	0.947	0.0235	0.0235	0.0233	0.0235
	0.5	0.946	0.946	0.946	0.947	0.0390	0.0390	0.0389	0.0392
	0.75	0.951	0.951	0.949	0.952	0.0580	0.0580	0.0576	0.0582
	1	0.948	0.947	0.945	0.947	0.0767	0.0767	0.0763	0.0769
	1.5	0.947	0.949	0.941	0.947	0.1144	0.1144	0.1136	0.1145
	2	0.954	0.953	0.952	0.948	0.1552	0.1552	0.1538	0.1552
	2.5	0.949	0.948	0.948	0.947	0.2019	0.2019	0.2003	0.2021

I. Coverage Probability

Table 1 displays the simulation results for the CP, whereas Figure 3 visually represents the results. The parameter value θ has a minor impact on the CP of all CIs. This indicates that the value of CP of all CIs is relatively constant, regardless of the value of the parameter θ. The sample size significantly affects the CP across all CPs. For small sample sizes (n = 10, 20, and 30), the CPs for the likelihood-based and Wald-type CIs are close to the nominal level of 0.95, whereas the bootstrap-t and BCa bootstrap CIs provide CPs that are noticeably less than 0.95. However, their performance improves with larger sample sizes, with the CPs approaching the nominal level more closely. This implies that although these approaches are sample size-dependent, they provide sufficient coverage for larger samples. Furthermore, the likelihood-based and Wald-type CIs exhibit a more rapid convergence rate in comparison to the bootstrap-t and BCa bootstrap CIs.

The likelihood-based and Wald-type CIs show greater stability in CP when both sample size and parameter value are considered, as they maintain values that are more closely approximate to the nominal level in a range of parameter values and sample sizes. On the other hand, the bootstrap-t and BCa bootstrap CIs demonstrate greater variability in CP, particularly for small sample sizes, in which case they tend to underperform.

II. Average Length

The AL usually decreases as the sample size increases, as expected in the evaluation of CIs. For example, when the sample size is n = 10 and θ = 2, the AL for the Wald-type CI is high, roughly 1.1303. Nevertheless, when the sample size is increased to n = 500, the AL for the Wald-type CI reduces significantly to approximately 0.1552.

The AL also varies with different parameter values of θ. As the value of the parameter θ increases, the AL tends to increase for all CIs. At θ = 0.2 and n = 10, the AL for the Wald-type CI is approximately 0.1132. However, at θ = 2.5, the AL increases to approximately 1.4856.

The Wald-type and likelihood-based CIs have very similar ALs for all parameter values when compared to the two other methods. This means that both Wald-type and likelihood-based CIs have a very similar interval width and coverage probability. The bootstrap-t and BCa bootstrap CIs tend to yield shorter intervals for lower values of θ, while demonstrating an increase in the

AL as the parameter value θ increases. The bootstrap-t CI generally provides the shortest interval. For example, for $\theta = 1$ and $n = 50$, the ALs are 0.2434 for likelihood-based CI, 0.2433 for Wald-type CI, 0.2370 for bootstrap-t CI, and 0.2393 for BCa bootstrap CI. These findings indicate that the bootstrap-t method yields the narrowest interval on average, but the likelihood-based method yields slightly wider intervals.

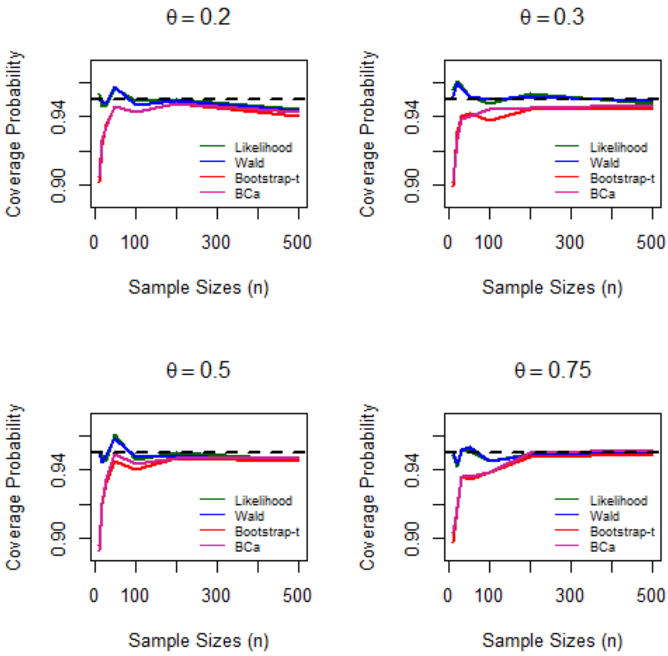

Figure 3: Plots of the CPs of the CIs for θ of the Iwueze distribution

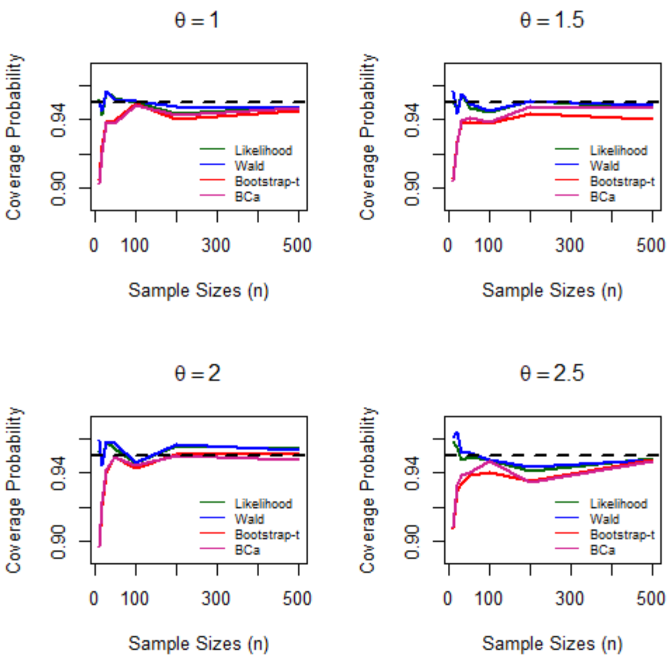

Figure 3: (Continued)

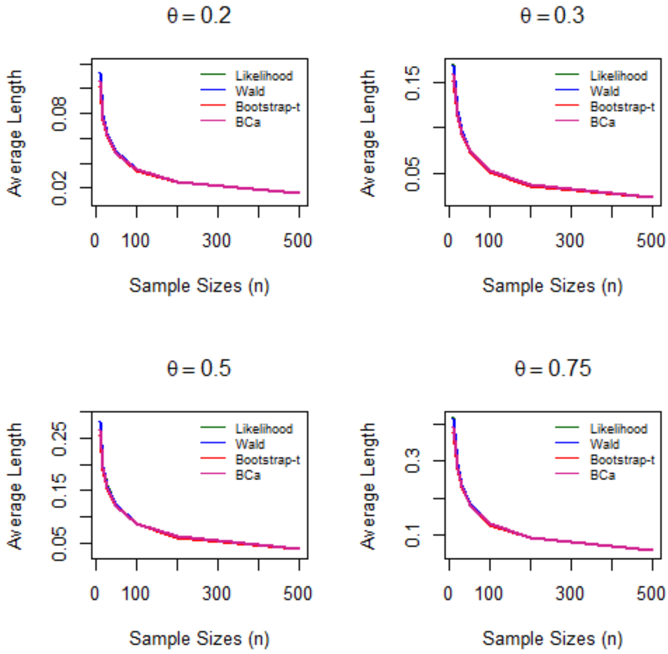

Figure 4: Plots of the ALs of the CIs for θ of the Iwueze distribution

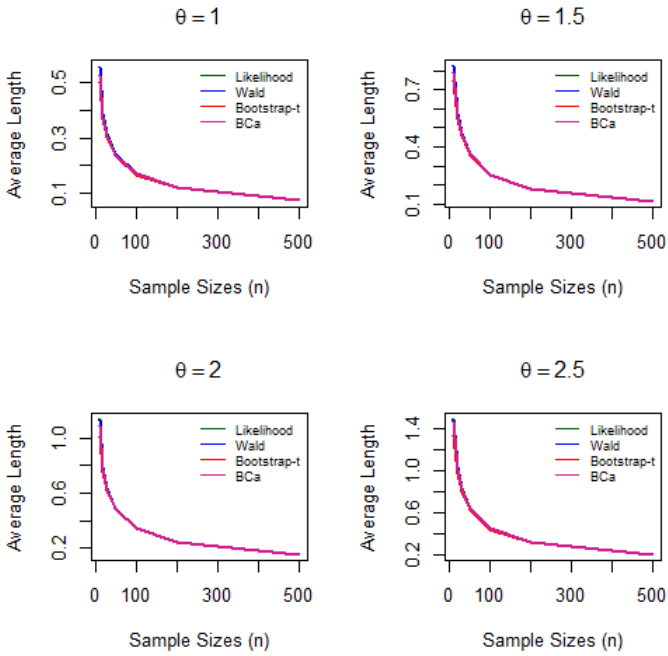

Figure 4: (Continued)

IV. Numerical Examples

We applied four CIs for the parameter of the Iwueze distribution defined in the previous section to two real-world situations. The adequacy of the Iwueze distribution's performance is being compared to that of the following alternative distributions:

- The Komal distribution [29]. Its pdf is

$$f(x;\theta)=\frac{\theta^2}{\theta^2+\theta+1}(1+\theta+x)e^{-\theta x},\ \ x>0,\theta>0.$$

- The Adya distribution [30]. Its pdf is

$$f(x;\theta)=\frac{\theta^3}{\theta^4+2\theta^2+2}(\theta+x)^2 e^{-\theta x},\ \ x>0,\theta>0.$$

- The Pranav distribution [31]. Its pdf is

$$f(x;\theta)=\frac{\theta^4}{\theta^4+6}(\theta+x^3)e^{-\theta x},\ \ x>0,\theta>0.$$

- The Prakaamy distribution [32]. Its pdf is

$$f(x;\theta)=\frac{\theta^6}{\theta^5+120}(1+x^5)e^{-\theta x},\ \ x>0,\theta>0.$$

- The Akshaya distribution [33]. Its pdf is

$$f(x;\theta)=\frac{\theta^4}{\theta^3+3\theta^2+6\theta+6}(1+x)^3 e^{-\theta x},\ \ x>0,\theta>0.$$

- The Rani distribution [34]. Its pdf is

$$f(x;\theta)=\frac{\theta^5}{\theta^5+24}(\theta+x^4)e^{-\theta x},\ \ x>0,\theta>0.$$

- The Rama distribution [35]. Its pdf is

$$f(x;\theta)=\frac{\theta^4}{\theta^3+6}(1+x^3)e^{-\theta x},\ \ x>0,\theta>0.$$

- The Suja distribution [36]. Its pdf is

$$f(x;\theta)=\frac{\theta^5}{\theta^4+24}(1+x^4)e^{-\theta x},\ \ x>0,\theta>0.$$

- The Ishita distribution [37]. Its pdf is

$$f(x;\theta)=\frac{\theta^3}{\theta^3+2}(\theta+x^2)e^{-\theta x},\ \ x>0,\theta>0.$$

- The Sujatha distribution [9]. Its pdf is

$$f(x;\theta)=\frac{\theta^3}{\theta^2+\theta+2}(1+x+x^2)e^{-\theta x},\ \ x>0,\theta>0.$$

- The Garima distribution [10](Shanker, 2016b). Its pdf is

$$f(x;\theta)=\frac{\theta}{\theta+2}(1+\theta+\theta x)e^{-\theta x},\ \ x>0,\theta>0.$$

- The Aradhana distribution [38]. Its pdf is

$$f(x;\theta)=\frac{\theta^3}{\theta^2+2\theta+2}(1+x)^2 e^{-\theta x},\ \ x>0,\theta>0.$$

- The Devya distribution [39]. Its pdf is

$$f(x;\theta)=\frac{\theta^5}{\theta^4+\theta^3+2\theta^2+6\theta+24}(1+x+x^2+x^3+x^4)e^{-\theta x},\ \ x>0,\theta>0.$$

- The Amarendra distribution [40]. Its pdf is

$$f(x;\theta)=\frac{\theta^4}{\theta^3+\theta^2+2\theta+6}(1+x+x^2+x^3)e^{-\theta x},\ \ x>0,\theta>0.$$

- The Shanker distribution [6]. Its pdf is

$$f(x;\theta)=\frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x},\ \ x>0,\theta>0.$$

- The Akash distribution [7]. Its pdf is

$$f(x;\theta)=\frac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x},\ \ x>0,\theta>0.$$

- The Lindley distribution [3]. Its pdf is

$$f(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x},\;\; x>0, \theta>0.$$

- The exponential distribution. Its pdf is

$$f(x;\theta)=\theta e^{-\theta x},\;\; x>0, \theta>0.$$

I. Lifetime Data about the Duration of Relief in the Analgesic Patients

The first data set consists of the lifetime data about the duration of relief (measured in minutes) experienced by 20 patients who were administered an analgesic. This data was reported by Gross and Clark [41]. The data are as follows: 1.1, 1.5, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0. Some descriptive statistics of the data set are reported in Table 2.

Table 2. *The descriptive statistics of the lifetime data about the duration of relief in the analgesic patients*

Min	Mean	Median	SD	Q1	Q3	Max
1.100	1.900	1.700	0.704	1.475	2.050	4.100

The ML technique is used to estimate all distribution parameters. For model comparison, we evaluated the log-likelihood (log L), Akaike's information criterion (AIC), and Bayesian information criterion (BIC). For this data set, estimates of the parameters, their standard errors (SE), and goodness of fit measures are given in Table 3.

Table 3. *The ML estimates, SE, AIC and BIC for the lifetime data about the duration of relief in the analgesic patients*

Distributions	Estimates (SE)	Log L	AIC	BIC
Iwueze	1.8013 (0.0312)	-25.9446	**53.8892**	**54.8849**
Komal	0.7404 (0.0146)	-31.1797	64.3593	65.3550
Adya	1.0602 (0.0146)	-28.4095	58.8189	59.8147
Pranav	1.4014 (0.0156)	-31.1933	64.3865	65.3823
Prakaamy	2.2735 (0.0261)	-30.7198	63.4396	64.4353
Akshaya	1.4417 (0.0282)	-26.5071	55.0141	56.0098
Rani	1.7195 (0.0163)	-32.6543	67.3085	68.3043
Rama	1.5213 (0.0232)	-29.8533	61.7066	62.7023
Suja	1.8954 (0.0248)	-30.2010	62.4020	63.3978
Ishita	1.0948 (0.0148)	-30.0824	62.1647	63.1604
Sujatha	1.1367 (0.0224)	-28.7488	59.4975	60.4933
Garima	0.7396 (0.0197)	-31.6058	65.2116	66.2073
Aradhana	1.1232 (0.0233)	-28.1850	58.3700	59.3658
Devya	1.8419 (0.0286)	-27.2522	56.5044	57.5001
Amarendra	1.4808 (0.0258)	-27.8193	57.6387	58.6344
Shanker	0.8039 (0.0142)	-29.8917	61.7833	62.7791
Akash	1.1569 (0.0212)	-29.7613	61.5226	62.5183
Lindley	0.8039 (0.0142)	-30.2536	62.5073	63.5030
Exponential	0.5263 (0.0139)	-32.8371	67.6742	68.6699

The AIC and BIC values in Table 3 illustrate that the Iwueze distribution provides an adequate fit to as compared with other distributions. The ML estimator for this data is 1.8013. Table 4 presents the 95% CIs for the parameter of the Iwueze distribution. The likelihood-based method yields a CI ranging from 1.4783 to 2.1720, with an interval length of 0.6937. Similarly, the

Wald-type method provides a CI of 1.4553 to 2.1472, also with a length of 0.6919, which is almost identical to the likelihood-based method in terms of range and uncertainty. In contrast, the bootstrap-t method and the BCa bootstrap method both produce notably narrower CIs.

Table 4. *The 95% CIs and lengths for the lifetime data about the duration of relief in the analgesic patients*

Methods	Confidence intervals	Lengths
Likelihood-based	(1.4783, 2.1720)	0.6937
Wald-type	(1.4553, 2.1472)	0.6919
Bootstrap-t	(1.6363, 2.0162)	0.3799
BCa Bootstrap	(1.5776, 1.9507)	0.3731

II. The Strengths of Glass Fibers

The second data set is from Smith and Naylor [42] on the strengths of 1.5 centimeter glass fibers measured at the National Physical Laboratory in England. This data set is given as follows: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89. Some descriptive statistics of the data set are reported in Table 5.

Table 5. *The descriptive statistics of the strengths of glass fibers*

Min	Mean	Median	SD	Q1	Q3	Max
0.550	1.507	1.590	0.3241	1.375	1.685	2.240

The ML method was utilized for estimating the parameters of the distributions. We assessed the log-likelihood (log L), Akaike's information criterion (AIC), and Bayesian information criterion (BIC) for model comparison. Table 6 provides estimates of the parameters, their standard errors (SE), and goodness of fit measures for this data set.

Table 6. *The ML estimates, SE, AIC and BIC for the strengths of glass fibers*

Distributions	Estimates (SE)	Log L	AIC	BIC
Iwueze	2.0894 (0.0137)	-68.6897	**139.3794**	**141.5225**
Komal	0.8905 (0.0070)	-84.5918	171.1836	173.3268
Adya	1.2237 (0.0064)	-77.1008	156.2016	158.3447
Pranav	1.5607 (0.0063)	-90.4814	182.9627	185.1059
Prakaamy	2.4974 (0.0100)	-93.0292	188.0583	190.2015
Akshaya	1.7091 (0.0130)	-69.5206	141.0413	143.1844
Rani	1.8802 (0.0063)	-98.1100	198.2199	200.3630
Rama	1.7313 (0.0098)	-84.8598	171.7197	173.8628
Suja	2.1133 (0.0099)	-88.7335	179.4670	181.6101
Ishita	1.2520 (0.0064)	-84.1406	170.2812	172.4243
Sujatha	1.3501 (0.0104)	-77.4048	156.8096	158.9527
Garima	0.9157 (0.0096)	-85.0308	172.0617	174.2048
Aradhana	1.3464 (0.0110)	-74.9384	151.8768	154.0199
Devya	2.1013 (0.0121)	-74.9042	151.8085	153.9516
Amarendra	1.7201 (0.0114)	-75.5186	153.0372	155.1804
Shanker	0.9563 (0.0066)	-81.1391	164.2781	166.4213
Akash	1.3554 (0.0096)	-81.8636	165.7272	167.8704

Lindley	0.9563 (0.0066)	-81.3693	164.7387	166.8818
Exponential	0.6636 (0.0070)	-88.8303	179.6606	181.8038

The AIC and BIC values, estimates of the parameters, their SEs, and measures of goodness of fit for this dataset are provided in Table 6. It shows that the Iwueze distribution fits better than other distributions. For this set of data, the ML estimator is 2.0894. Table 7 reports comparisons of 95% CIs and their lengths for parameter estimation using several methods. The likelihood-based method estimates the CI to be between 1.8691 and 2.3292, with an interval length of 0.4601. The Wald-type method yields a marginally narrower CI, ranging from 1.8597 to 2.3192, and has an interval length of 0.4595, closely aligning with the results of the likelihood-based method. In contrast, the bootstrap-t method offers a narrower CI, spanning from 2.0240 to 2.1552, with the shortest interval length of 0.1312. Similarly, the BCa bootstrap method provides an even tighter CI, ranging from 2.0288 to 2.1627, with the interval length at 0.1339.

Table 7. *The 95% CIs and lengths for the strengths of glass fibers*

Methods	Confidence intervals	Lengths
Likelihood-based	(1.8691, 2.3292)	0.4601
Wald-type	(1.8597, 2.3192)	0.4595
Bootstrap-t	(2.0240, 2.1552)	0.1312
BCa Bootstrap	(2.0288, 2.1627)	0.1339

Conclusion and Discussion

This paper proposes and evaluates four approaches for using likelihood-based, Wald-type, bootstrap-t, bias-corrected and accelerated (BCa) bootstrap methods to construct confidence intervals (CIs) for the parameter of the Iwueze distribution. This study also derived and provided the explicit formula for the Wald-type CI. The evaluation of CIs in simulation studies involves the consideration of both the coverage probability (CP) and the average length (AL) of the intervals. As the sample sizes increase, the results indicate a notable pattern where the CPs of all methods converge toward the nominal confidence level. The likelihood-based and Wald-type CIs yielded satisfactory outcomes in terms of coverage probabilities, even for the setting of small sample sizes. However, the bootstrap-t and BCa bootstrap CIs provide the CP less than the nominal confidence level, especially in small sample sizes. The practical application of all CIs was shown by applying them to medical and engineering data, producing results consistent with the simulation study's results.

The bootstrap techniques examined in this study rely on the assumption that resampled data accurately represent the underlying population. For datasets with very small sample sizes and significant skewness, the validity of the assumption that resampled data accurately represent the underlying population may be compromised. Consequently, this could impact the reliability of the CIs derived from these methods. Moreover, the computational requirements of bootstrap techniques, particularly the BCa bootstrap CI, might present challenges in situations where computational resources are limited. To facilitate the computation of bootstrap confidence intervals in the R programming language, numerous packages are accessible, with the 'boot' package [43] and the 'bootstrap' package [44] being notable examples.

Future research could explore other mixed distributions, such as the Chris-Jerry distribution [45], Hamza distribution [46], among others. The construction of CIs for the coefficient of variation and the population mean is an interesting topic that requires additional research. Additionally, there appears to be a gap in the literature regarding hypothesis testing for the parameters of the

Iwueze distribution. These topics represent valuable opportunities for future studies.

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