BAYESIAN INFERENCE OF WEIBULL-PARETO DISTRIBUTION UNDER DOUBLE TYPE I HYBRID CENSORED DATA

Khawla Boudjerda

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Departement of mathematics, Mohammed Seddik Ben Yahia university, Jijel, Algeria
khawla.boudjerda@univ-jijel.dz

Abstract

This paper investigates the estimation of parameters, reliability, and failure rate functions of the Weibull-Pareto distribution using double type I hybrid censored data. We begin by applying the maximum likelihood method to derive point estimates for the distribution parameters. Subsequently, we explore Bayesian estimation techniques, obtaining Bayesian estimators under various loss functions to enhance robustness. To compute these estimators, we utilize Markov Chain Monte Carlo (MCMC) methods, facilitating effective sampling from complex posterior distributions. We employ Pitman closeness criteria to compare the performance of Bayesian estimators against those derived from maximum likelihood estimation, providing a comprehensive evaluation of their accuracy and efficiency. Additionally, a real data example is presented to illustrate the practical application of our methodologies. The results underscore the advantages of the Bayesian approach, particularly in scenarios characterized by hybrid censoring, while also contributing to the broader understanding of reliability analysis in statistical modeling.

Keywords: Weibull-Pareto distribution, double type I hybrid censored data, MCMC methods, Pitman closeness.

1. Introduction

Historically, it has been believed that the Pareto distribution and its associated generalizations are appropriate for modeling income and wealth distributions. A notable generalization is the New Weibull-Pareto distribution (NWPD), defined by [1]. This distribution is particularly useful in modeling real-life scenarios and can simulate data with a bathtub-shaped hazard rate, which is important for feature engineering in reliability analysis. The cumulative distribution function (CDF) of the NWPD, characterized by the shape parameter β and the scale parameters α and λ is given by:

$$F(x;\alpha,\beta,\lambda) = 1 - exp(-\alpha(\frac{x}{\lambda})^{\beta}), \;\; x > 0, \alpha, \beta, \lambda > 0, \tag{1}$$

its probability density function (PDF) is given

$$f(x;\alpha,\beta,\lambda) = \frac{\alpha\beta}{\lambda}(\frac{x}{\lambda})^{\beta-1} exp(-\alpha(\frac{x}{\lambda})^{\beta}), \;\; x > 0. \tag{2}$$

The Reliability and the failure rate function of the NWPD are respectively given by

$$R(x;\alpha,\beta,\lambda) = 1 - F(x;\alpha,\beta,\lambda) = exp(-\alpha(\frac{x}{\lambda})^{\beta}), \;\; x > 0, \tag{3}$$

and

$$h(x;\alpha,\beta,\lambda) = \frac{f(x;\alpha,\beta,\lambda)}{R(x;\alpha,\beta,\lambda)} = \frac{\alpha\beta}{\lambda}(\frac{x}{\lambda})^{\beta-1}, \;\; x > 0. \tag{4}$$

The representation graphics of the PDF and reliability functions of the NWPD for differents values of α, β and λ are given by

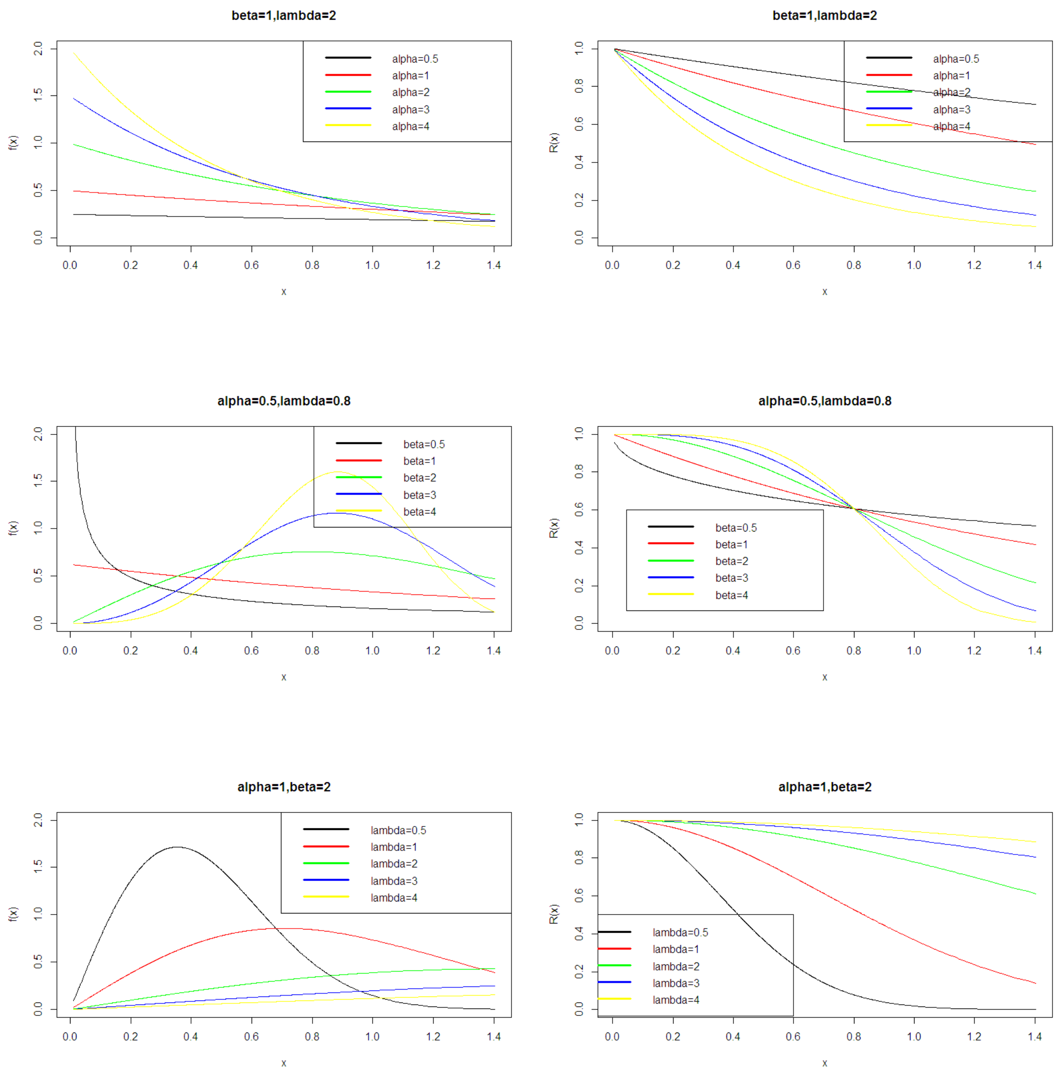

Figure 1: *The PDF and reliability functions of the NWPD for differents values of α, β and λ*

Many authors have discussed various properties of the NWPD and have obtained the maximum likelihood estimators for its parameters (see, [2], [1], [3]).

In this article, we obtain the maximum likelihood and Bayesian estimators of the parameters, as well as the reliability and failure rate functions, using double type I hybrid censored data. The hybrid censored scheme (HCS) is the mixture of type I and type II censored test has appeared. In double type I hybrid censored sheme, we place n units identically independent distributed in a test. k is un integer verify $k > 0$ and $k < n$ where k is given at the beginning. Let t_1 and t_2 be the censored time checked $0 < t_1 < t_2$. The test carried out until time t_1 and k_1 are the number of units which fail. If $k_1 > k$, the test will end at time t_1 and $n - k_1$ units are removed from the test. If $k_1 < k$, the test will end at the time t_2 and k_2 is the number of failed components. The two previous cases can be summarized as follows:

Case 1: $X_{1;n}$, $X_{2;n}$,...,$X_{k_1;n}$ if $k_1 > k$.
Case 2: $X_{1;n}$, $X_{2;n}$,...,$X_{k_2;n}$ if $k_1 < k$.

Hybrid censored data appears in many works. The two-sided confidence interval using double Type I hybrid censored schemes is introduced by [4]. The exact confidence bounds for an exponential parameter under hybrid censoring data are proposed by [5]. Then, [6] obtained the exact two-sided confidence interval of θ following the approach of [5]. Recently, hybrid censored schemes have gained significant popularity in reliability and life-testing experiments. The estimation of parameters for the log-normal distribution based on hybrid censored data is studied by [7]. Additionally, the estimation and prediction of the generalized Lindley distribution using hybrid censored data are provided by [8], while the estimation of unknown parameters of the inverted linear exponential distribution based on Type I hybrid censored data is investigated by [6].

The reste of this paper is organaised as follows, In section 2, we estimate the parameters, reliability and failure rate functions of the NWPD distribution using the maximum likelihood mathod. Bayesian estimation under different loss functions is presented in section 3. Monte-Carlo simulation results are applied in section 5. Analysis of one real data set is introduced in section 6.

2. Maximum likelihood estimation

In this section, maximum likelihood estimators of parameters, reliability function and failure rate function of the NWPD based on double type I hybrid censored data is considered. Let $x = (x_{1:n}, x_{2:n}, ..., x_{m,n})$ be double type I hybrid censored sample from the NWPD, then the likelihood function is given by

$$L(\alpha, \beta, \lambda|x) = \prod_{i=1}^{m} f(x_{i:n}; \alpha, \beta, \lambda)[1 - F(t; \alpha, \beta, \lambda)]^{n-m}, \tag{5}$$

where

$$t = \begin{cases} t_1, & case\, 1 \\ t_2, & case\, 2 \end{cases}$$

and

$$m = \begin{cases} k_1, & case\, 1 \\ k_2, & case\, 2 \end{cases}$$

$$L(\alpha, \beta, \lambda|x) = \alpha^m \beta^m \lambda^{-m\beta} \prod_{i=1}^{m} x_i^{\beta-1} exp[-\alpha \sum_{i=1}^{m} (\frac{x_i}{\lambda})^\beta - \alpha(n-k)(\frac{t}{\lambda})^\beta]. \tag{6}$$

Then the log-likelihood function is

$$l(\alpha, \beta, \lambda|x) = mln\alpha + mln\beta - m\beta ln\lambda + (\beta - 1) \sum_{i=1}^{m} lnx_i - \alpha \sum_{i=1}^{m} (\frac{x_i}{\lambda})^\beta - \alpha(n-m)(\frac{t}{\lambda})^\beta. \tag{7}$$

The log-likelihood function derivatives are given by

$$\begin{cases} \frac{\partial l(\alpha,\beta,\lambda|x)}{\partial \alpha} = \frac{m}{\alpha} - \sum_{i=1}^{m} (\frac{x_i}{\lambda})^\beta - (n-m)(\frac{t}{\lambda})^\beta. \\ \frac{\partial l(\alpha,\beta,\lambda|x)}{\partial \beta} = \frac{m}{\beta} - mln\lambda + \sum_{i=1}^{m} lnx_i - \alpha \sum_{i=1}^{m} ln(\frac{x_i}{\lambda})(\frac{x_i}{\lambda})^\beta - \alpha(n-m)ln(\frac{t}{\lambda})(\frac{t}{\lambda})^\beta. \\ \frac{\partial l(\alpha,\beta,\lambda|x)}{\partial \lambda} = -\frac{m\beta}{\lambda} + \frac{\alpha\beta}{\lambda}(\frac{x_i}{\lambda})^\beta + \frac{\alpha\beta}{\lambda}(n-m)(\frac{t}{\lambda})^\beta. \end{cases}$$

We can't obtain the explicit analytical form of the estimators, several methods are applied to find the approximate values of the maximum likelihood estimators $\widehat{\alpha}_{ml}$, $\widehat{\beta}_{ml}$ and $\widehat{\lambda}_{ml}$. In this article, we use the R package BB ([10]), (the BBsolve function).

To obtain the maximum likelihood estimators of the reliability function $R(x, \alpha, \beta, \lambda)$ and the failure rate function $h(x, \alpha, \beta, \lambda)$ we replace α, β and λ by $\widehat{\alpha}_{ml}$, $\widehat{\beta}_{ml}$ and $\widehat{\lambda}_{ml}$ in the formulas 3 et 4.

3. Bayesian estimation

In this section, we consider the Bayesian estimators of parameters, reliability function and failure rate function of the NWPD based on double type I hybrid censored data. We suppose, that the parameters α, β and λ of the NWPD have independent gamma distributions:

$$\pi(\alpha) \propto \alpha^{a_1-1} exp(-b_1\alpha), \ \ \alpha > 0. \tag{8}$$

$$\pi(\beta) \propto \beta^{a_2-1} exp(-b_2\beta), \ \ \beta > 0. \tag{9}$$

$$\pi(\lambda) \propto \lambda^{a_3-1} exp(-b_3\alpha), \ \ \lambda > 0. \tag{10}$$

where the hyper parameters $a_1 > 0$, $b_1 > 0$, $a_2 > 0$, $b_2 > 0$, $a_3 > 0$ and $b_3 > 0$.
Thus, the joint prior distribution of (α, β, λ) is given by

$$\pi(\alpha, \beta, \lambda) \propto \alpha^{a_1-1}\beta^{a_2-1}\lambda^{a_3-1} exp(-b_1\alpha - b_2\beta - b_3\lambda)$$

Using the Bayesian formula, the posterior probability density function of (α, β, λ) is

$$\pi(\alpha, \beta, \lambda|x) = K^{-1}\alpha^{m+a_1-1}\beta^{m+a_2-1}\lambda^{-m\beta+a_3-1}\zeta(x)\phi(x) \tag{11}$$

where K is the normalisation constant given by

$$K = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \alpha^{m+a_1-1}\beta^{m+a_2-1}\lambda^{-m\beta+a_3-1}\zeta(x)\phi(x)d\alpha d\beta d\lambda.$$

$\zeta(x) = \prod_{i=1}^{m} x_i^{\beta-1}$ and $\phi(x) = exp[-\alpha\sum_{i=1}^{m}(\frac{x_i}{\lambda})^\beta - \alpha(n-m)(\frac{t}{\lambda})^\beta - b_1\alpha - b_2\beta - b_3\lambda]$

3.1. Bayesian estimation under squared loss function

Under squared loss function $L_1(\theta, \delta) = (\theta - \delta)^2$, the Bayesian estimators is the posterior mean

$$\widehat{\delta}_{B1} = E_\pi(\delta|x) = \int\int\int \delta\pi(\alpha, \beta, \lambda|x)d\alpha d\beta d\lambda \tag{12}$$

In the case of the NWPD, we obtain the Bayesian estimators of the parameter α, β and λ

$$\begin{aligned}\hat{\alpha}_{B1} &= E_\pi(\alpha|x)\\ &= \int\int\int \alpha\pi(\alpha, \beta, \lambda|x)d\alpha d\beta d\lambda\\ &= K^{-1}\int\int\int_0^{+\infty} \alpha^{m+a_1}\beta^{m+a_2-1}\lambda^{-m\beta+a_3-1}\zeta(x)\phi(x)d\alpha d\beta d\lambda\end{aligned}$$

$$\begin{aligned}\hat{\beta}_{B1} &= E_\pi(\beta|x)\\ &= \int\int\int \beta\pi(\alpha, \beta, \lambda|x)d\alpha d\beta d\lambda\\ &= K^{-1}\int\int\int_0^{+\infty} \alpha^{m+a_1-1}\beta^{m+a_2}\lambda^{-m\beta+a_3-1}\zeta(x)\phi(x)d\alpha d\beta d\lambda\end{aligned}$$

$$\begin{aligned}\hat{\lambda}_{B1} &= E_\pi(\lambda|x)\\ &= \int\int\int \lambda\pi(\alpha, \beta, \lambda|x)d\alpha d\beta d\lambda\\ &= K^{-1}\int\int\int_0^{+\infty} \alpha^{m+a_1-1}\beta^{m+a_2-1}\lambda^{-m\beta+a_3}\zeta(x)\phi(x)d\alpha d\beta d\lambda.\end{aligned}$$

The Bayesian estimator of the reliability function under the squared loss function is given by

$$\widehat{R}_{B1}(x) = E_\pi(R(x)|x) = \int\int\int R(x)\pi(\alpha, \beta, \lambda|x)d\alpha d\beta d\lambda$$

$$= K^{-1}\int\int\int_0^{+\infty}\alpha^{m+a_1-1}\beta^{m+a_2-1}\lambda^{-m\beta+a_3-1}\zeta(x)\phi(x)exp\left[-\alpha\left(\frac{x}{\lambda}\right)^{\beta}\right]d\alpha d\beta d\lambda. \quad (13)$$

The Bayesian estimator of the failure rate function under the squared loss function is given by

$$\widehat{h}_{B1}(x) = E_\pi(h(x)|x) = \int\int\int h(x)\pi(\alpha,\beta,\lambda|x)d\alpha d\beta d\lambda$$

$$= K^{-1}\int\int\int_0^{+\infty}\alpha^{m+a_1}\beta^{m+a_2}\lambda^{-m\beta+a_3-2}\zeta(x)\left(\frac{x}{\lambda}\right)^{\beta-1}\phi(x)d\alpha d\beta d\lambda. \quad (14)$$

3.2. Bayesian estimation under entropy loss function

Under Entropy loss function $L_2(\theta,\delta) = (\frac{\delta}{\theta})^p - plog(\frac{\delta}{\theta}) - 1$, the Bayesian estimator is given by

$$\widehat{\delta}_{B_2} = E_\pi(\delta^{(-p)}|x)^{\frac{-1}{p}} = \left[\int\int\int\delta^{-p}\pi(\alpha,\beta,\lambda|x)d\alpha d\beta d\lambda\right]^{\frac{-1}{p}}, \;\; p\neq 0 \quad (15)$$

In the case of the NWPD, we obtained

$$\begin{aligned}\hat{\alpha}_{B2} &= E_\pi(\alpha^{-p}|x)^{\frac{-1}{p}}, \;\; p\neq 0\\ &= \left[\int\int\int\alpha^{-p}\pi(\alpha,\beta,\lambda|x)d\alpha d\beta d\lambda\right]^{\frac{-1}{p}}\\ &= \left[K^{-1}\int\int\int_0^{+\infty}\alpha^{m+a_1-1-p}\beta^{m+a_2-1}\lambda^{-m\beta+a_3-1}\zeta(x)\phi(x)d\alpha d\beta d\lambda\right]^{\frac{-1}{p}}.\end{aligned}$$

$$\begin{aligned}\hat{\beta}_{B2} &= E_\pi(\beta^{-p}|x)^{\frac{-1}{p}}, \;\; p\neq 0\\ &= \left[\int\int\int\beta^{-p}\pi(\alpha,\beta,\lambda|x)d\alpha d\beta d\lambda\right]^{\frac{-1}{p}}\\ &= \left[K^{-1}\int\int\int_0^{+\infty}\alpha^{m+a_1-1}\beta^{m+a_2-1-p}\lambda^{-m\beta+a_3-1}\zeta(x)\phi(x)d\alpha d\beta d\lambda\right]^{\frac{-1}{p}}.\end{aligned}$$

$$\begin{aligned}\hat{\lambda}_{B2} &= E_\pi(\lambda^{-p}|x)^{\frac{-1}{p}}, \;\; p\neq 0\\ &= \left[\int\int\int\lambda^{-p}\pi(\alpha,\beta,\lambda|x)d\alpha d\beta d\lambda\right]^{\frac{-1}{p}}\\ &= \left[K^{-1}\int\int\int_0^{+\infty}\alpha^{m+a_1-1}\beta^{m+a_2-1}\lambda^{-m\beta+a_3-1-p}\zeta(x)\phi(x)d\alpha d\beta d\lambda\right]^{\frac{-1}{p}}.\end{aligned}$$

The Bayesian estimator of the reliability function under the entropy loss function is given by

$$\widehat{R}_{B2}(x) = E_\pi(R(x)^{-p}|x)^{\frac{-1}{p}} = \left[\int\int\int R(x)^{-p}\pi(\alpha,\beta,\lambda|x)d\alpha d\beta d\lambda\right]^{\frac{-1}{p}}, \;\; p\neq 0$$

$$= \left[K^{-1}\int\int\int_0^{+\infty}\alpha^{k+a_1-1}\beta^{k+a_2-1}\lambda^{-m\beta+a_3-1}\zeta(x)\phi(x)exp\left[p\alpha(\frac{x}{\lambda})^{\beta}\right]d\alpha d\beta d\lambda\right]^{\frac{-1}{p}}. \quad (16)$$

The Bayesian estimator of the failure rate function under the entropy loss function is given by

$$\widehat{h}_{B2}(x) = E_\pi(h(x)^{-p}|x)^{\frac{-1}{p}} = \left[\int\int\int h(x)^{-p}\pi(\alpha,\beta,\lambda|x)d\alpha d\beta d\right]^{\frac{-1}{p}}, \;\; p\neq 0$$

$$= \left[K^{-1}\int\int\int_0^{+\infty}\alpha^{k+a_1-p-1}\beta^{k+a_2-p-1}\lambda^{-m\beta+a_3+p-1}\zeta(x)(\frac{x}{\lambda})^{p(\beta-1)}\phi(x)d\alpha d\beta d\lambda\right]^{\frac{-1}{p}}. \quad (17)$$

3.3. Bayesian estimation under Linex loss function

Under Linex loss function $L_3(\delta,\theta) = exp(a(\delta-\theta)) - (\delta-\theta) - 1$, the Bayesian estimator is given by

$$\widehat{\delta}_{B_3} = \frac{-1}{a} log E_\pi(exp(-a\delta)) = -\frac{1}{a} log\left[\int\int\int exp(-a\delta)\pi(\alpha,\beta,\lambda|x)d\alpha d\beta d\lambda\right],\ a\neq 0 \qquad (18)$$

In the case of the NWPD, the Bayesian estimators of the parameters are

$$\begin{aligned}\hat{\alpha}_{B3} &= \frac{-1}{a} log E_\pi(exp(-a\alpha)),\ a\neq 0\\ &= \frac{-1}{a} log\left[\int\int\int exp(-a\alpha)\pi(\alpha,\beta,\lambda|x)d\alpha d\beta d\lambda\right]\\ &= \frac{-1}{a} log\left[K^{-1}\int\int\int_0^{+\infty} exp(-a\alpha)\alpha^{m+a_1-1}\beta^{m+a_2-1}\lambda^{-m\beta+a_3-1}\zeta(x)\phi(x)d\alpha d\beta d\lambda\right].\end{aligned}$$

$$\begin{aligned}\hat{\beta}_{B3} &= \frac{-1}{a} log E_\pi(exp(-a\beta)),\ a\neq 0\\ &= \frac{-1}{a} log\left[\int\int\int exp(-a\beta)\pi(\alpha,\beta,\lambda|x)d\alpha d\beta d\lambda\right]\\ &= \frac{-1}{a} log\left[K^{-1}\int\int\int_0^{+\infty} exp(-a\beta)\alpha^{m+a_1-1}\beta^{m+a_2-1}\lambda^{-m\beta+a_3-1}\zeta(x)\phi(x)d\alpha d\beta d\lambda\right].\end{aligned}$$

$$\begin{aligned}\hat{\lambda}_{B3} &= \frac{-1}{a} log E_\pi(exp(-a\lambda)),\ a\neq 0\\ &= \frac{-1}{a} log\left[\int\int\int exp(-a\lambda)\pi(\alpha,\beta,\lambda|x)d\alpha d\beta d\lambda\right]\\ &= \frac{-1}{a} log\left[K^{-1}\int\int\int_0^{+\infty} exp(-a\lambda)\alpha^{m+a_1-1}\beta^{m+a_2-1}\lambda^{-m\beta+a_3-1}\zeta(x)\phi(x)d\alpha d\beta d\lambda\right].\end{aligned}$$

The Bayesian estimator of the reliability function under Linex loss function is given by

$$\widehat{R}_{B3}(x) = -\frac{1}{a} log E_\pi(exp(-aR(x))) = -\frac{1}{a} log\left[\int\int\int exp(-aR(x))\pi(\alpha,\beta,\lambda|x)d\alpha d\beta d\lambda\right],\ a\neq 0$$

$$= \frac{-1}{a} log\left[K^{-1}\int\int\int_0^{+\infty}\alpha^{k+a_1-1}\beta^{k+a_2-1}\lambda^{-m\beta+a_3-1}\zeta(x)\phi(x)exp(-a exp(-\alpha(\frac{x}{\lambda})^{\beta}))d\alpha d\beta d\lambda\right]. \qquad (19)$$

The Bayesian estimator of the failure rate function under the Linex loss function is given by

$$\widehat{h}_{B3}(x) = -\frac{1}{a} log E_\pi(exp(-ah(x))) = -\frac{1}{a}\left[\int\int\int exp(-ah(x))\pi(\alpha,\beta,\lambda|x)d\alpha d\beta d\lambda\right],\ a\neq 0$$

$$= \frac{-1}{a} log\left[K^{-1}\int\int\int_0^{+\infty} exp\left(\frac{-a\alpha\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}\right)\alpha^{m+a_1-1}\beta^{m+a_2-1}\lambda^{-m\beta+a_3-1}\zeta(x)\phi(x)d\alpha d\beta d\lambda\right] \qquad (20)$$

We can't obtain the explicit analytical form of all these estimators, for this we will use the MCMC methods in the next section.

4. Simulation

In this section, we generate $N = 1000$ samples of differents sizes $(n = 20, 40, 50)$ of the NWPD. We pose the $\alpha = \beta = \lambda = 1$, then we use the R package called BB (especially the function BBsolve) to solve the system of equations and find the maximum likelihood estimators and their mean squared errors, the results are in the next table

Table 1: *Maximum likelihood estimators of the parameters, reliability function and the failure rate function (mean squared errors)*

(n,m)	(t_1, t_2)	$\widehat{\alpha}_{ml}(MSE)$	$\widehat{\beta}_{ml}(MSE)$	$\widehat{\lambda}_{ml}(MSE)$	$\widehat{R}_{ml}(MSE)$	$\widehat{h}_{ml}(MSE)$
(20,16)	(6,7)	0.9216	0.8325	1.0967	0.5108	0.7456
		(0.0016)	(0.0280)	(0.0093)	(0.0014)	(0.0647)
	(8,11)	0.9333	0.7632	1.1417	0.5077	0.6896
		(0.0043)	(0.0560)	(0.0200)	(0.0012)	(0.0963)
	(10,15)	0.9570	0.6774	1.1749	0.4935	0.6378
		(0.0018)	(0.1040)	(0.0306)	(0.0004)	(0.1311)
(40,34)	(20,25)	0.9981	0.6733	1.1490	0.4728	0.6723
		$3.4 * 10^{-6}$	(0.1067)	(0.0222)	$(2.7 * 10^{-7})$	(0.1073)
	(22,30)	1.0054	0.6481	1.1590	0.4728	0.6723
		$(3 * 10^{-5})$	(0.1238)	(0.0259)	$(1.5 * 10^{-5})$	(0.1188)
	(24,33)	1.0041	0.6330	1.1702	0.4687	0.6395
		$(1.7 * 10^{-5})$	(0.1346)	(0.0289)	$(1.2 * 10^{-5})$	(0.1299)
(50,40)	(25,32)	1.0061	0.6712	1.1436	0.4686	0.6784
		$(3.8 * 10^{-5})$	(0.1080)	(0.0208)	$(1.4 * 10^{-5})$	(0.1034)
	(30,35)	1.0099	0.6511	1.1579	0.4671	0.6608
		$(9.9 * 10^{-5})$	(0.1217)	(0.0249)	$(2.7 * 10^{-5})$	(0.1150)
	(32,38)	1.0087	0.6420	1.1665	0.4678	0.6502
		$(7.6 * 10^{-5})$	(0.1281)	(0.0277)	$(2 * 10^{-5})$	(0.1222)

We consider the hyper parameters of the prior distribution

$$a_1 = a_2 = a_3 = 1, \quad b_1 = b_2 = b_3 = 2.$$

We present the Bayesian estimators of the parameters, the reliability function, the failure funtion and their posterior risks under the squared loss function, the entropy loss function and Linex loss function in the followings tables

Table 2: *Bayesian estimators of the parameters, reliability function and the failure rate function under squared loss function (posterior risk)*

(n,m)	(t_1, t_2)	$\widehat{\alpha}_{B_1}(PR)$	$\widehat{\beta}_{B_1}(PR)$	$\widehat{\lambda}_{B_1}(PR)$	$\widehat{R}_{B_1}(PR)$	$\widehat{h}_{B_1}(PR)$
(20,16)	(6,7)	1.0199 (0.0022)	0.9702 (0.0027)	1.0497 (0.0043)	0.4889 (0.0002)	0.9500 (0.0024)
	(8,11)	1.0213 (0.0022)	0.9716 (0.0025)	1.0511 (0.0043)	0.4891 (0.0002)	0.9510 (0.0023)
	(10,15)	1.0208 (0.0021)	0.9711 (0.0025)	1.0506 (0.0042)	0.4890 (0.0002)	0.9508 (0.0024)
(40,34)	(20,25)	1.0217 (0.0025)	0.9721 (0.0028)	1.0506 (0.0047)	0.4892 (0.0002)	0.9510 (0.0023)
	(22,30)	1.0221 (0.0027)	0.9725 (0.0030)	1.0519 (0.0049)	0.4892 (0.0002)	0.9510 (0.0023)
	(24,33)	1.0221 (0.0029)	0.9724 (0.0032)	1.0519 (0.0051)	0.4892 (0.0002)	0.9508 (0.0024)
(50,40)	(25,32)	1.0213 (0.0025)	0.9717 (0.0029)	1.0510 (0.0047)	0.4892 (0.0002)	0.9507 (0.0024)
	(30,35)	1.0235 (0.0029)	0.9739 (0.0030)	1.0533 (0.0051)	0.4895 (0.0002)	0.9520 (0.0022)
	(32,38)	1.0255 (0.0031)	0.9759 (0.0031)	1.0553 (0.0055)	0.4898 (0.0003)	0.9532 (0.0021)

Table 3: *Bayesian estimators of the parameters, reliability function and the failure rate function under entropy loss function (posterior risk)*

(n,m)	(t_1, t_2)	$\widehat{\alpha}_{B_2}(PR)$	$\widehat{\beta}_{B_2}(PR)$	$\widehat{\lambda}_{B_2}(PR)$	$\widehat{R}_{B_2}(PR)$	$\widehat{h}_{B_2}(PR)$
(20,16)	(6,7)	1.0182 (0.0181)	0.9685 (0.0319)	1.0481 (0.0469)	0.4795 (0.0151)	0.9493 (0.0520)
	(8,11)	1.0197 (0.0195)	0.9699 (0.0304)	1.0495 (0.0488)	0.4796 (0.0153)	0.9504 (0.0508)
	(10,15)	1.0192 (0.0191)	0.9695 (0.0309)	1.0491 (0.0479)	0.4796 (0.0151)	0.9501 (0.0511)
(40,34)	(20,25)	1.0198 (0.0196)	0.9701 (0.0303)	1.0496 (0.0484)	0.4798 (0.0156)	0.9502 (0.0510)
	(22,30)	1.0201 (0.0199)	0.9703 (0.300)	1.0499 (0.0487)	0.4798 (0.0157)	0.9502 (0.0003)
	(24,33)	1.0199 (0.0197)	0.9701 (0.0302)	1.0497 (0.0485)	0.4799 (0.0159)	0.9499 (0.0532)
(50,40)	(25,32)	1.0194 (0.0192)	0.9697 (0.0307)	1.0492 (0.0480)	0.4797 (0.0155)	0.9499 (0.0513)
	(30,35)	1.0214 (0.0212)	0.9717 (0.0286)	1.0512 (0.0499)	0.4800 (0.0161)	0.9512 (0.0500)
	(32,38)	1.0233 (0.0230)	0.9736 (0.0267)	1.0531 (0.0517)	0.4803 (0.0166)	0.9523 (0.0488)

Table 4: *Bayesian estimators of the parameters, reliability function and the failure rate function under Linex loss function (posterior risk)*

(n,m)	(t_1, t_2)	$\widehat{\alpha}_{B_3}(PR)$	$\widehat{\beta}_{B_3}(PR)$	$\widehat{\lambda}_{B_3}(PR)$	$\widehat{R}_{B_3}(PR)$	$\widehat{h}_{B_3}(PR)$
(20,16)	(6,7)	1.0190	0.9693	1.0488	0.4796	0.9496
		(0.0008)	(0.0008)	(0.0008)	(0.0093)	(0.0003)
	(8,11)	1.0204	0.9707	1.0502	0.4797	0.9507
		(0.0008)	(0.0008)	(0.008)	(0.0094)	(0.0003)
	(10,15)	1.0200	0.9703	1.0498	0.4796	0.9505
		(0.0008)	(0.0008)	(0.0008)	(0.0093)	(0.0003)
(40,35)	(20,25)	1.0207	0.9710	1.0505	0.4798	0.9506
		(0.0010)	(0.0010)	(0.0010)	(0.0094)	(0.0003)
	(22,30)	1.0210	0.9714	1.0508	0.4799	0.9506
		(0.0011)	(0.0011)	(0.0010)	(0.0094)	(0.0003)
	(24,33)	1.0209	0.9712	1.0507	0.4800	0.9504
		(0.0011)	(0.0011)	(0.0011)	(0.0094)	(0.0004)
(50,40)	(25,32)	1.0203	0.9707	1.0500	0.4798	0.9503
		(0.0010)	(0.0010)	(0.0010)	(0.0093)	(0.0003)
	(30,35)	1.0224	0.9728	1.0521	0.4801	0.9516
		(0.0011)	(0.0011)	(0.0011)	(0.0094)	(0.0004)
	(32,38)	1.0243	0.9747	1.0541	0.4803	0.9528
		(0.0012)	(0.0012)	(0.0012)	(0.0094)	(0.0004)

Comparaison of estimators

We use [9] criterion to make a comparaison between the Bayesian and the maximum likelihood estimators.

Definition 1. An estimator $\widehat{\delta}_1$ of a parameter θ is better than author estimator $\widehat{\delta}_2$ in the sens of Pitman's criterion if for all values of θ

$$P(|\widehat{\delta}_1 - \theta| < |\widehat{\delta}_2 - \theta|) > \frac{1}{2}$$

We conclude that for all values of (n, m, t_1, t_2), the Bayesian estimators of the parametrs α, β and λ are better than the maximum likelihood estimators in the case of the NWPD.

Table 5: *Pitman criteriom*

(n,m)	(t1,t2)	parameters	squared	entropy	Linex
$(20,16)$	$(6,7)$	α	0.848	0.855	0.855
		β	0.897	0.902	0.899
		λ	0.920	0.926	0.921
	$(8,11)$	α	0.843	0.844	0.804
		β	0.891	0.904	0.897
		λ	0.907	0.910	0.909
	$(10,15)$	α	0.798	0.809	0.804
		β	0.865	0.877	0.872
		λ	0.883	0.890	0.887
$(50,40)$	$(25,32)$	α	0.885	0.887	0.886
		β	0.883	0.918	0.906
		λ	0.881	0.885	0.881
	$(22,30)$	α	0.876	0.881	0.878
		β	0.891	0.902	0.897
		λ	0.881	0.884	0.882
	$(24,33)$	α	0.880	0.887	0.886
		β	0.907	0.913	0.910
		λ	0.878	0.893	0.881
$(50,40)$	$(25,32)$	α	0.885	0.887	0.886
		β	0.899	0.908	0.904
		λ	0.870	0.872	0.871
	$(30,35)$	α	0.911	0.917	0.915
		β	0.901	0.913	0.905
		λ	0.898	0.902	0.901
	$(35,38)$	α	0.900	0.906	0.903
		β	0.898	0.903	0.900
		λ	0.905	0.911	0.909

Discussion of results

We notice that all the probabilities are greater than 0.5. So, the Bayesian estimators of the parameters of the NWPD are more consistent than its maximum likelihood estimators.

5. Real data analysis

In this section, we use the real data represents the runoff amounts at Jug Bridge, Maryland

Table 6: *Data set of the rumoff amounts at Jug Bridge, Maryland*

0.17	0.19	0.23	0.33	0.39	0.39	0.40	0.45	0.52	0.56	0.59	0.64	0.66
0.70	0.76	0.77	0.78	0.95	0.97	1.02	1.12	1.24	1.59	1.74	2.92	

- We apply the Kolmogorov-Smirnov (K-S) test on this data set, we obtain

Table 7: *K-S test*

K-S value	0.1601
P-value	0.9062

So, we observe that the NWPD model fit quite well to this data set.
Using this real data set, we calculate the differents estimators of the parameters, reliability and failure rate functions with the maximum likelihood and the Bayesian methods, the results are:

Table 8: *Maximum likelihood estimators*

m	t_1	t_2	$\widehat{\alpha}_{ml}$	$\widehat{\beta}_{ml}$	$\widehat{\lambda}_{ml}$	$\widehat{R}_{ml}$	$\widehat{h}_{ml}$
15	0.8	1.2	1.0014	0.9842	1.0969	0.5021	0.9039
			$(2 * 10^{-6})$	(0.0024)	(0.0094)	(0.0008)	(0.0092)
	1	1.8	1.0963	0.9641	0.9665	0.4237	1.1036
			(0.0092)	(0.0012)	(0.0011)	(0.0023)	(0.0107)
18	0.9	1.2	1.1100	0.9241	0.9736	0.4182	1.0767
			(0.0121)	(0.0054)	(0.0007)	(0.0029)	(0.0059)
	1	1.5	1.0963	0.9641	0.09665	0.4237	1.1036
			(0.0092)	(0.0012)	(0.0011)	(0.0023)	(0.0107)
20	1.1	1.6	1.0915	0.9802	0.9673	0.4271	1.1117
			(0.0083)	(0.0004)	(0.0010)	(0.0020)	(0.0124)
	1.5	1.9	1.0816	0.9924	0.9764	0.4349	1.1015
			(0.0066)	$(5 * 10^{-5})$	(0.0005)	(0.0013)	(0.0103)
23	1.3	1.7	1.0801	1.0087	0.9671	0.4330	1.1198
			(0.0064)	$(1 * 10^{-5})$	0.001	0.0015	0.0123
	1.6	2	1.0643	0.9859	0.9644	0.4357	1.0919
			(0.0041)	(0.0002)	(0.0013)	(0.0013)	(0.0084)

Table 9: *Bayesian estimators of the parameters α, β and λ*

m	t_1	t_2	$\widehat{\alpha}_{B_1}$	$\widehat{\beta}_{B_1}$	$\widehat{\lambda}_{B_1}$	$\widehat{\alpha}_{B_2}$	$\widehat{\beta}_{B_2}$	$\widehat{\lambda}_{B_2}$	$\widehat{\alpha}_{B_3}$	$\widehat{\beta}_{B_3}$	$\widehat{\lambda}_{B_3}$
15	0.8	1.2	1.1644	1.1158	1.1547	1.0598	1.0101	1.0498	1.0768	1.0262	1.0668
			(0.0888)	(0.0717)	(0.0850)	(0.1438)	(0.1443)	(0.1439)	(-0.0740)	(-0.0265)	(0.0617)
	1	1.8	1.2226	1.1727	1.2126	1.0801	1.0303	1.0701	1.1131	1.0634	1.1032
			(0.0797)	(0.0599)	(0.0753)	(0.1586)	(0.1590)	(0.1587)	(0.1072)	(0.0615)	(0.0982)
18	0.9	1.2	1.2547	1.2048	1.2447	1.1332	1.0833	1.1232	1.1446	1.0963	1.1349
			(0.0921)	(0.0692)	(0.0871)	(0.1055)	(0.1066)	(0.1056)	(0.1351)	(0.0919)	(0.1266)
	1	1.5	1.2435	1.1937	1.2335	1.1639	1.1140	1.1539	1.1769	1.1271	1.1670
			(0.0928)	(0.0710)	(0.0881)	(0.0748)	(0.0753)	(0.0749)	(0.1629)	(0.1197)	(0.1544)
20	1.1	1.6	1.2507	1.2011	1.2408	1.1732	1.1238	1.1633	1.1838	1.1339	1.1738
			(0.1009)	(0.0784)	(0.0960)	(0.0655)	(0.0656)	(0.0654)	(0.1687)	(0.1257)	(0.1603)
	1.5	1.9	1.2581	1.2083	1.2408	1.1351	1.0854	1.1252	1.1769	1.1271	1.1670
			(0.1025)	(0.0791)	(0.0974)	(0.1036)	(0.1040)	(0.1037)	(0.1629)	(0.1197)	(0.1544)
23	1.3	1.7	1.2360	1.1863	1.2260	1.1460	1.0966	1.1361	1.1964	1.1466	1.1864
			(0.0952)	(0.0741)	(0.0906)	(0.0927)	(0.0896)	(0.0899)	(0.1793)	(0.1368)	(0.1709)
	1.6	2	1.2294	1.1795	1.2194	1.1772	1.1273	1.1672	1.2037	1.1541	1.1938
			(0.0904)	(0.0700)	(0.0859)	(0.0615)	(0.0620)	(0.0616)	(0.1874)	(0.1433)	(0.1771)

Table 10: *Bayesian estimators of the reliability and failure rate functions*

m	t_1	t_2	$\hat{R}_{B_1}$	$\hat{h}_{B_1}$	$\hat{R}_{B_2}$	$\hat{h}_{B_2}$	$\hat{R}_{B_3}$	$\hat{h}_{B_3}$
15	0.8	1.2	0.5012	1.0586	0.4903	0.9725	0.5004	0.9720
			(0.0008)	(0.0034)	(0.0096)	(0.0774)	(-0.0577)	(0.0283)
	1	1.8	0.4966	1.0486	0.4911	0.8969	0.4923	0.9755
			(0.0005)	(0.0023)	(0.0088)	(0.0630)	(-0.0413)	(0.0248)
18	0.9	1.2	0.4950	1.0511	0.4973	1.0252	0.4919	1.0130
			(0.0005)	(0.0026)	(0.0026)	(0.0247)	(-0.0406)	(-0.0129)
	1	1.5	0.5067	1.0502	0.4914	1.0194	0.4893	1.0011
			(0.0012)	(0.0025)	(0.0085)	(0.0305)	(-0.0353)	(-0.0011)
20	1.1	1.6	0.4946	1.0458	0.4950	1.0130	0.4924	1.0058
			(0.0004)	(0.0021)	(0.0003)	(0.0169)	(-0.0342)	(-0.0045)
	1.5	1.9	0.4947	1.0453	0.4950	1.0130	0.4924	1.0058
			(0.0005)	(0.0020)	(0.0049)	(0.0369)	(-0.0415)	(-0.0057)
23	1.3	1.7	0.4913	1.0376	0.4931	1.0019	0.4917	1.0033
			(0.0003)	(0.0014)	(0.0068)	(0.0480)	(-0.0402)	(-0.0032)
	1.6	2	0.4975	1.0526	0.5019	0.0330	0.4940	1.0170
			1(0.0006)	(0.0027)	(-0.0019)	(0.0169)	(-0.0449)	(-0.0169)

6. Conclusion

In this article, we have estimated the parameters and reliability functions of the Weibull-Pareto model, a valuable tool in engineering applications. We employed two estimation methods: the classical maximum likelihood method and the Bayesian approach, utilizing double type I hybrid censored data. Due to the complexity of the classical case, we relied on numerical methods to derive the estimators, as explicit analytical forms were not attainable. For the Bayesian method, we assumed a gamma prior distribution for the parameters and calculated estimators under squared, entropy, and Linex loss functions. Since the estimators remained in integral form, we applied the Metropolis-Hastings algorithm to obtain their numerical values.

Our comparative analysis using Pitman's criterion revealed that the Bayesian estimators consistently outperformed the maximum likelihood estimators. To illustrate the applicability of our methodologies, we also conducted a study using real data. The findings underscore the effectiveness of the Weibull-Pareto model and highlight the advantages of the Bayesian approach in reliability analysis, providing valuable insights for future research and practical applications.

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