ALPHA POWER TRANSFORMED WEIBULL LOMAX DISTRIBUTION: PROPERTIES AND ITS APPLICATIONS

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Abstract

We proposed a new model called the Alpha Power Transformed Weibull-Lomax (APTWL) distribution which extends the Weibull Lomax distribution and have an increasing, decreasing and bathtub shapes for the hazard rate function. Various structural properties of the new distribution are derived including moments, probability weighted moments, generating and quantile function. The Renyi and q entropies are also obtained. Statistical inference is presented for the APTWL distribution using the method of maximum likelihood estimation to estimate the parameters of proposed distribution. The potentiality of the new model is illustrated by means of three real life datasets. The results of the analysis of the datasets show the superiority of APTWL distribution over some compared distributions.

Keywords: Alpha power transformation, Hazard function, Likelihood estimation, Lomax distribution, Weibull Lomax distribution

1. Introduction

The Lomax distribution also known as the Pareto Type II distribution is a continuous distribution that is used to model extreme events that follow a power law distribution. It is named after K.S Lomax [12] who first introduced. It is a heavy tailed distribution. That is, the extreme events are more likely to occur in other distributions like the normal distribution. It is often used to model phenomena like city sizes, earthquake magnitudes and financial returns. The Lomax distributions has applications in various fields including income and wealth inequality, acturial science, biological and medical sciences, engineering, longevity and reliability modeling, economics, insurance and geology. Overall, the Lomax distribution is a useful tool for modeling and analyzing extreme events.

Various authors extend the classical distributions to make them applicable in various fields. For example, Marshall-Olkin Extended Lomax (MOEL) was introduced by Ghitany *et al.* [7], Exponentiated Lomax (EL) was studied by Abdul-Moniem and Abdel-Hameed [1], Beta Lomax (BL) was examined by Lemonte and Cordeiro [11], Gamma Lomax (GL) was presented by Cordeiro *et al.* [4], Power Lomax (PL) was studied by Rady *et al.* [17] and Weibull Power Lomax(WPL) distribution was introduced by Hussain *et al.* [8]. The Weibull Lomax (WL) distribution was recently introduced and its mathematical and statistical features were examined by Tahir *et al.* [19].

Weibull-Lomax(WL) distribution which is introduced by Tahir *et al* [19] which extends the Lomax distribution has increasing and decreasing shapes for the hazard rate function. It has wider application in areas such as engineering, survival and lifetime data, hydrology and economics

(income inequality).
The cdf of WL distribution is given by,

$$F(x) = 1 - exp\left\{-\alpha\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^{\alpha} - 1\right\}^{b}\right\}\cdot \tag{1}$$

The pdf of WL distribution is given by,

$$f(x) = \frac{ab\alpha}{\beta}[1+\frac{x}{\beta}]^{b\alpha-1}\left\{1-[1+\frac{x}{\beta}]^{-\alpha}\right\}^{b-1}\exp\left\{-\alpha([1+\frac{x}{\beta}]^{-\alpha}-1)^{b}\right\}\cdot \tag{2}$$

where $x > 0$; $a > 0$ and $b > 0$ are two parameters.

We are motivated to do this work for developing distribution which follow increasing, decreasing and constant failure rates. The main aim of this study is to provide another extension of the Weibull Lomax distribution introduced by Tahir *et al.*[19] using the Alpha power transformation defined by Mahdavi and Kundu [13].

Mahdavi and Kundu [13] proposed a new class of distributions called the Alpha Power Transformation (APT) family. It is introduced to analyse lifetime data obtained from systems that exhibit variety of monotonic and non-monotonic failure patterns.
The cdf of the APT family is defined by,

$$\begin{aligned} F_{APT}(x) &= \frac{\alpha^{G(x)}-1}{\alpha-1}, if\ \alpha > 0, \alpha \neq 1 \\ &= G(x), if\ \alpha = 1\cdot \end{aligned} \tag{3}$$

The corresponding pdf of APT family is,

$$\begin{aligned} f_{APT}(x) &= \frac{log\,\alpha}{\alpha-1}\,\alpha^{G(x)}g(x), if\ \alpha > 0, \alpha \neq 1 \\ &= g(x), if\ \alpha = 1\cdot \end{aligned} \tag{4}$$

2. Alpha Power Transformed Weibull Lomax Distribution

2.1. Probability Density Function (pdf) and Cumulative Distribution Function (cdf)

The Alpha Power Transformed Weibull Lomax (APTWL) distribution is obtained by using Weibull Lomax distribution as baseline distribution in alpha power transformation which is proposed by Mahdavi and Kundu [13]. Inserting equation (1) in (3), we get the five parameter APTWL cdf which is given by,

$$F(x) = \frac{\alpha^{1-exp\left\{-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^{\theta}-1\right\}^{b}\right\}}-1}{\alpha-1}\cdot \tag{5}$$

And the corresponding pdf is obtained by inserting (2) in (4) and is given by,

$$f(x) = \frac{log\,\alpha}{\alpha-1}\,\frac{ab\theta}{\beta}\,\left(1+\frac{x}{\beta}\right)^{b\theta-1}\,\left\{1-\left(1+\frac{x}{\beta}\right)^{-\theta}\right\}^{b-1}exp\left\{-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^{\theta}-1\right\}^{b}\right\}\alpha^{1-exp\left\{-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^{\theta}-1\right\}^{b}\right\}}\cdot \tag{6}$$

where $x > 0$; $\alpha > 0$ is the additional shape parameter.

Figure 1 and Figure 2 provides plots of the pdf for some selected values of parameters. It is clear from the graph that the APTWL densities appears to be right Skewed and flexible heavy tailed distribution.

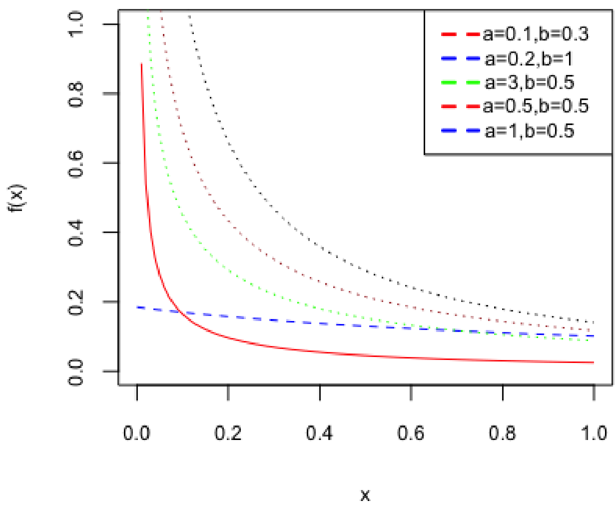

Figure 1: *pdf plot for* $\alpha = 0.5, \beta = 0.75, \theta = 0.5$

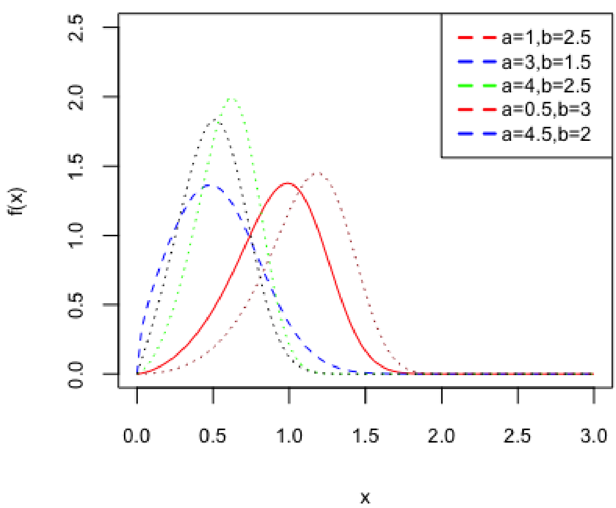

Figure 2: *pdf plot for* $\alpha = 3, \beta = 5, \theta = 4$

2.2. Quantile Function

Quantile functions are used in statistical analysis to summarize distributions and are essential in constructing box plots, quantile-quantile plots and in statistical methods (Mood *et.al.*) [16].

Quantile Function can be obtained by taking the inverse of the cumulative distribution function of a distribution. For $q \in (0,1)$,the quantile function of x is obtained by,

$$x_q = \beta \left\{ \left(\left[-\frac{1}{a} \log \left\{ 1 - \left\{ (\log \alpha)^{-1} \log \left[q(\alpha-1)+1 \right] \right\} \right\} \right]^{\frac{1}{b}} + 1 \right)^{\frac{1}{\theta}} - 1 \right\}. \tag{7}$$

The root of equation (7) which is x_q, gives the unique solution for every value of $q \in (0,1)$ for a particular combination of parameter values of (α,β,θ,a,b). If q = 0.5, the median of APTWL distribution denoted by $x_{0.5}$ can be obtained from equation (7) with its expression given as

$$x_{0.5} = \beta \left\{ \left(\left[-\frac{1}{a} \log \left\{ 1 - \left\{ (\log \alpha)^{-1} \log \left[0.5\,(\alpha-1)+1 \right] \right\} \right\} \right]^{\frac{1}{b}} + 1 \right)^{\frac{1}{\theta}} - 1 \right\}. \tag{8}$$

2.3. Reliability Function

The reliability function is a fundamental concept in reliability engineering used to model and analyze the longevity and performance of systems and components over time. It is crucial for determining maintenance schedules, warranty analysis, and risk assessment(Elsayed) [6]. Reliability function can be obtained from,

$$\begin{aligned} R(x) &= 1 - F(x) \\ &= (\alpha-1)^{-1} \left[\alpha - \alpha^{1-exp\left\{ -a\left\{ \left[1+\left(\frac{x}{\beta}\right) \right]^{\theta} - 1 \right\}^{b} \right\}} \right]. \end{aligned} \tag{9}$$

2.4. Hazard Function

The hazard function provides insights into the risk of failure at any given time point which helps to understand the dynamics of the failure process. It is a crucial concept in survival analysis and reliability engineering (Kalbfleisch and Prentice) [9]. The hazard function is the probability of failure in an infinitely small time period between x and x + dx given that the subject has survived up to time x. The hazard function is defined by,

$$\begin{aligned} h(x) &= \frac{f(x)}{R(x)} \\ &= \frac{\frac{\log \alpha}{\alpha-1}\,\frac{ab\theta}{\beta}\left(1+\frac{x}{\beta}\right)^{b\theta-1}\left\{1-\left(1+\frac{x}{\beta}\right)^{-\theta}\right\}^{b-1} exp\left\{-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^{\theta}-1\right\}^{b}\right\}}{(\alpha-1)^{-1}} \\ &\qquad \left[\frac{\alpha^{1-exp\left\{ -a\left\{ \left[1+\left(\frac{x}{\beta}\right) \right]^{\theta} - 1 \right\}^{b} \right\}}}{\left[\alpha - \alpha^{1-exp\left\{ -a\left\{ \left[1+\left(\frac{x}{\beta}\right) \right]^{\theta} - 1 \right\}^{b} \right\}} \right]} \right]. \end{aligned} \tag{10}$$

Figure 3 and Figure 4 shows that hazard rate shapes can take different shapes such as constant, increasing and decreasing shape.

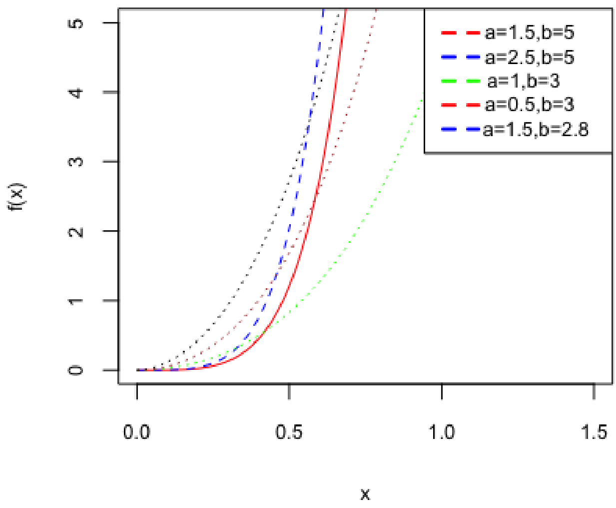

Figure 3: *hrf plot for* $\alpha = 1.5, \beta = 1.5, \theta = 1.5$

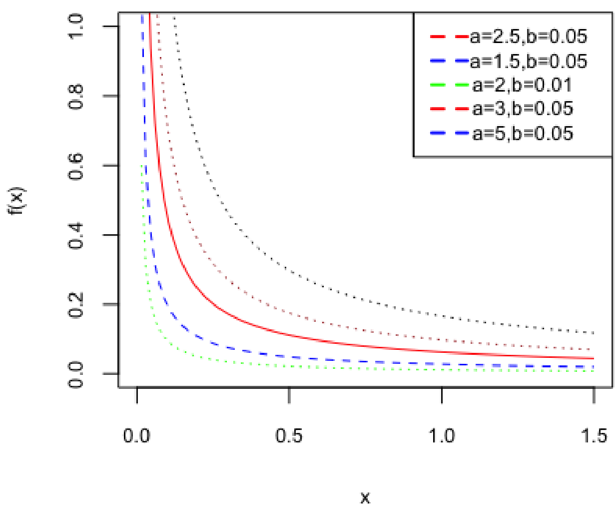

Figure 4: *hrf plot for* $\alpha = 3$, $\beta = 0.5$, $\theta = 1.5$

3. Properties

3.1. Moments

Moments are applied in various statistical methods including parameter estimation, hypothesis testing and in the method of moments for deriving estimators (DeGroot and Schervish) [5]. The rth moment of a random variable X is obtained as follows,

$$\mu_r' = E[X^r]$$

$$\mu_r^{'} = \int_0^{\infty} x^r \frac{\log\alpha}{\alpha-1}\,\frac{ab\theta}{\beta}\,\left(1+\frac{x}{\beta}\right)^{b\theta-1}\,\left\{1-\left(1+\frac{x}{\beta}\right)^{-\theta}\right\}^{b-1} exp\left\{-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^{\theta}-1\right\}^{b}\right\}$$
$$\alpha^{1-exp\left\{-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^{\theta}-1\right\}^{b}\right\}}dx\cdot \qquad (11)$$

Using the power series expansion $\alpha^k = \sum_{j=0}^{\infty}\frac{(log\alpha)^j}{j!}k^j$ in equation (11) and then applying binomial expansion $(1-z)^{\beta} = \sum_{i=0}^{\infty}(-1)^i\,\binom{\beta}{i}\,z^i$ we get,

$$\mu_r^{'} = \sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+1}\,a^{k(ij+1)+2}\,b\theta\,(\log\alpha)^{j+1}}{i!\,k!\,(j-1)!\,\beta(\alpha-1)}\int_0^{\infty} x^r\,\left[\left(1+\frac{x}{\beta}\right)^{-\theta}\right]^{-\frac{1}{\theta}-b-bk(ij+1)}$$
$$\left[1-\left(1+\frac{x}{\beta}\right)^{-\theta}\right]^{(b-1)+bk(ij+1)}.$$

Put $y = \left(1+\frac{x}{\beta}\right)^{-\theta}$ and then integrating we get,

$$\mu_r^{'} = \sum_{m=0}^{r}\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+m+1}\,a^{k(ij+1)+2}\,\beta^r\,b(\log\alpha)^{j+1}\binom{r}{m}}{i!\,k!\,(j-1)!\,(\alpha-1)}$$
$$B(\frac{1}{\theta}\,(m-r)-b-bk(ij+1),b+bk(ij+1))\cdot \qquad (12)$$

3.2. Moment Generating Function

The moment generating function(mgf) of a random variable X is defined by $M_x(t) = E\left[e^{tx}\right]$, for values of t in some neighborhood of zero for which this expectation exists (DeGroot and Schervish) [5]. It is an alternate method for analyzing results instead of working directly with the pdf and cdf of a random variable X. We obtain the moment generating function of a random variable X of the APTWL distribution as,

$$\begin{aligned} M_X(t) &= \sum_{r=0}^{\infty}\frac{t^r}{r!}\,\mu_r^{'} \\ &= \sum_{m=0}^{r}\sum_{i,j,k,r=0}^{\infty}\frac{(-1)^{i+m+1}\,a^{k(ij+1)+2}\,\beta^{r-1}\,t^r b(\log\,)^{j+1}\binom{r}{m}}{i!\,k!\,r!\,(j-1)!(\alpha-1)} \\ &\qquad B(\frac{1}{\theta}(m-r)-b-bk(ij+1),b+bk(ij+1))\cdot \end{aligned} \qquad (13)$$

3.3. Probability Weighted Moments

Probability Weighted Moments (PWMs) are a set of statistical measures used to summarize the probability distribution of a random variable. It is used to derive estimators of the parameters and quantiles of generalized distributions. The (h,s)th PWM of X is given by,

$$\rho_{h,s} = E\left[X^h\, F(x)^s\right]$$
$$= \int_0^\infty x^h\, \frac{\log\alpha}{\alpha-1}\, \frac{ab\theta}{\beta}\, \left(1+\frac{x}{\beta}\right)^{b\theta-1} \left\{1-\left(1+\frac{x}{\beta}\right)^{-\theta}\right\}^{b-1} exp\left\{-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^{\theta}-1\right\}^{b}\right\}$$
$$\alpha^{1-exp\left\{-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^{\theta}-1\right\}^{b}\right\}} \left[\frac{\alpha^{1-exp\left\{-a\left\{\left[1+\left(\frac{x}{\beta}\right)\right]^{\theta}-1\right\}^{b}\right\}}-1}{\alpha-1}\right]^{s}. \tag{14}$$

Using the power series expansion and binomial expansion in equation (14), we get,

$$\rho_{h,s} = \sum_{i=0}^{s}\sum_{j,k=0}^{\infty}(-1)^{k+s-i}\binom{s}{i}\binom{j}{k}\frac{ab\theta(\log\alpha)^{j+1}(i+1)^j}{j!\,\beta\,(\alpha-1)^{s+1}}\int_0^\infty x^h\left(1+\frac{x}{\beta}\right)^{b\theta-1}$$
$$\left\{1-\left(1+\frac{x}{\beta}\right)^{-\theta}\right\}^{b-1} exp\left\{(k+1)\left\{-a\left(1+\frac{x}{\beta}\right)^{\theta}-1\right\}^{b}\right\}dx\cdot$$

Put $y=\left(1+\frac{x}{\beta}\right)^{-\theta}$ and then integrating, we get,

$$\rho_{h,s} = \sum_{i=0}^{s}\sum_{j,k,m,n=0}^{\infty}(-1)^{n+k+s-i}\binom{s}{i}\binom{j}{k}\binom{h}{m}\frac{a^{n+1}\,\beta^h\, b\,(\log\alpha)^{j+1}\,(i+1)^j\,(k+1)^n}{(\alpha-1)^{(s+1)}\, j!\, n!}$$
$$B\left(\frac{1}{\theta}(m-h)-b(1+n)-1\,, b(1+n)\right)\cdot \tag{15}$$

3.4. Renyi Entropy and q Entropy

The Renyi and q entropies are two important measures in information theory for examining the unpredictability associated with random variables that follow a lifetime distribution. The entropy of a random variable X is a measure of the uncertain variation.
The Renyi entropy is defined by,

$$\delta_R(\omega) = (1-\omega)^{-1}\log\left[\int_0^\infty (f(x))^\omega\, dx\right]\cdot$$

By applying the binomial expansion and exponential expansion in the pdf, we get,

$$\delta_R(\omega) = (1-\omega)^{-1}\log\left[a^\omega\left(\frac{b\theta\log\alpha}{\beta(\alpha-1)}\right)^\omega\sum_{i,j,k=0}^{\infty}(-1)^{i+k}\binom{j}{i}j^k\,\frac{(\omega\log\alpha)^i}{j!\,k!}\int_0^\infty\left(1+\frac{x}{\beta}\right)^{\omega(b\theta-1)}\right.$$
$$\left.\left\{1-\left(1+\frac{x}{\beta}\right)^{-\theta}\right\}^{\omega(b-1)}\left\{\left(1+\frac{x}{\beta}\right)-1\right\}^{bk}dx\cdot\right.$$

Put $y=\left(1+\frac{x}{\beta}\right)^{-\theta}$ and then integrating, we get,

$$\delta_R(\omega) = (1-\omega)^{-1}\log\left[a^{\omega+k}\,\beta^{1-\omega}\,\theta^{\omega-1}\left(\frac{b\log\alpha}{\alpha-1}\right)^\omega\sum_{i,j,k=0}^{\infty}(-1)^{i+k}\binom{j}{i}j^k\,\frac{(\omega\log\alpha)^j}{j!\,k!}\right.$$
$$\left. B\left(\frac{1}{\theta}(\omega-1)-b(\omega+k)\,, 1+bk+\omega(b-1)\right)\cdot\right. \tag{16}$$

The q entropy $H_q(f)$ is defined by,

$$H_q(f) = \frac{1}{q-1} log \left[1 - I_q(f)\right]$$
$$where\ I_q(f) = \int_R f^q(x)dx \cdot$$

$$H_q(f) = \frac{1}{q-1} log \Bigg[1 - a^{q+k} \beta^{1-q} \theta^{q-1} \left(\frac{b\, log\, \alpha}{\alpha - 1}\right)^q \sum_{i,j,k=0}^{\infty} (-1)^{i+k} \binom{j}{i} j^k \frac{(q\, log\, \alpha)^j}{j!\, k!}$$
$$B\left(\frac{1}{\theta}(q-1) - b(q+k), 1 + bk + q(b-1)\right) \cdot \quad (17)$$

3.5. Order Statistics

In a random sample of size n drawn from the APTWL distribution, we estimate the density of the ith order statistic $X_{i:n}$, say $f_{i:n}(x)$. We have (for i = 1,...,n),

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} F^{i-1}(x) \ [1 - F(x)]^{n-i} \ f(x) \quad (18)$$

Now we can write,

$$F(x)^{i+j-1} = \frac{1}{(\alpha - 1)^{i+j-1}} \left[\alpha^{\left\{1 - exp\left\{-a\left[\left(1+\frac{x}{\beta}\right)^{\theta} - 1\right]^b\right\}\right\}} - 1\right]^{i+j-1} \quad (19)$$

Using the binomial expansion $(1-z)^{\beta} = \sum_{i=0}^{\infty} (-1)^i \binom{\beta}{i} z^i$ in equation (18) and then applying power series expansion $\alpha^k = \sum_{j=0}^{\infty} \frac{(log \alpha)^j}{j!} k^j$ we get,

$$F(x)^{i+j-1} = \frac{1}{(\alpha - 1)^{i+j-1}} (-1)^{i+j+k-1} \sum_{m,k=0}^{\infty} \binom{i+j-1}{k} \frac{(log\ \alpha)^m}{m!} k^m \sum_{t=0}^{\infty} (-1)^t \binom{m}{t}$$
$$exp\left\{-at\left\{\left(1 + \frac{x}{\beta}\right)^{\theta} - 1\right\}^b\right\} \quad (20)$$

Substituting equation (20) in equation (18), we get,

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \frac{(log\ \alpha)^m\ k^m\ (at)^s}{m!\ \alpha^{i+j-1}} \sum_{j=0}^{n-i} \sum_{m,k,t,s=0}^{\infty} (-1)^{i+j+k+t+s-1} \binom{n-i}{j} \binom{i+j-1}{k}$$
$$\binom{m}{t} \left\{\left(1 + \frac{x}{\beta}\right)^{\theta} - 1\right\}^{bs} \quad (21)$$

4. Estimation

4.1. Maximum Likelihood Estimation

We consider the estimation of the unknown parameters of the APTWL distribution by the maximum likelihood method. The maximum likelihood approach is the most commonly used of the several parameter estimating techniques which have been validated in the literature. Let

$x_1, x_2, x_3, ..., x_n$ be a sample of size n from the APTWL distribution. The log likelihood (ll) function is given by

$$ll = n\,log\,(\frac{log\,\alpha}{\alpha-1}) \;+\; n\,log\,a \;+\; n\,log\,b \;+\; n\,log\,\theta \;-\; n\,log\,\beta \;+\; (b\theta-1)\sum_{i=1}^{n} log\,(1+\frac{x}{\beta}) + (b-1)\sum_{i=1}^{n} log\left\{1-(1+\frac{x}{\beta})^{-\theta}\right\} + \sum_{i=1}^{n}\left\{-a\left(1+\frac{x}{\beta}\right)^{\theta}-1\right\}^{b} + log\,\alpha\sum_{i=1}^{n}\left[1-exp\left\{-a\left(1+\frac{x}{\beta}\right)^{\theta}-1\right\}^{b}\right]. \quad (22)$$

Differentiating ll with respect to each parameter a,b,α,β and θ and setting the result equals to zero, we obtain maximum likelihood estimates (MLEs).

$$\frac{\partial ll}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n}\left[(1+\frac{x}{\beta})^{\theta}-1\right]^{b}\left\{1 \;+ log\,\alpha\; e^{[-a(1+\frac{x}{\beta})-1]^{b}}\right\}. \quad (23)$$

$$\frac{\partial ll}{\partial b} = \frac{n}{b} \;+\; \theta\sum_{i=1}^{n} log(1+\frac{x}{\beta}) \;+\sum_{i=1}^{n} log\left\{1-(1+\frac{x}{\beta})^{-\theta}\right\} - a\sum_{i=1}^{n}\left\{(1+\frac{x}{\beta})^{\theta}-1\right\}^{b}\left[log\left\{(1+\frac{x}{\beta})^{\theta}-1\right\} - log\;\alpha\; e^{-a\;\sum_{i=1}^{n}\left\{(1+\frac{x}{\beta})^{\theta}-1\right\}^{b}}\right]. \quad (24)$$

$$\frac{\partial ll}{\partial \alpha} = \frac{n}{\frac{log\,\alpha}{\alpha-1}}\;\frac{1}{\alpha} \;+\sum_{i=1}^{n}\left[1-log\;\alpha\; e^{-a\;\sum_{i=1}^{n}\left\{(1+\frac{x}{\beta})^{\theta}-1\right\}^{b}}\right]\frac{1}{\alpha} = \frac{1}{\alpha}\left[\frac{n(\alpha-1)}{log\,\alpha} \;+\sum_{i=1}^{n}\left(1-log\;\alpha\; e^{-a\;\sum_{i=1}^{n}\left\{(1+\frac{x}{\beta})^{\theta}-1\right\}^{b}}\right)\right]. \quad (25)$$

$$\frac{\partial ll}{\partial \beta} = -\frac{n}{\beta} - \frac{b\theta-1}{\beta^2}\sum_{i=1}^{n}\frac{x}{\left(1+\frac{x}{\beta}\right)^{3}} + \frac{\theta(b-1)}{\beta^2}\sum_{i=1}^{n}\frac{x(1+\frac{x}{\beta})^{-\theta-1}}{\left\{1-\left(1+\frac{x}{\beta}\right)^{-\theta}\right\}} - \frac{ab\theta}{\beta^2}\sum_{i=1}^{n}\left\{-a\left(1+\frac{x}{\beta}\right)^{\theta}-1\right\}^{b-1} x\left(1+\frac{x}{\beta}\right)^{\theta-1}\left\{1-log\sum_{i=1}^{n} exp\left(-a(1+\frac{x}{\beta})^{\theta}-1\right)^{b}\right\}. \quad (26)$$

$$\frac{\partial ll}{\partial \theta} = \frac{n}{\theta} \;+\; b\sum_{i=1}^{n} log(1+\frac{x}{\beta}) \;+ b-1)\;\theta\sum_{i=1}^{n}(1+\frac{x}{\beta})^{\theta}\; log\left\{(1+\frac{x}{\beta})^{\theta}\right\} - a\,b\sum_{i=1}^{n}\left\{(1+\frac{x}{\beta})^{\theta}-1\right\}^{b-1}(1+\frac{x}{\beta})^{\theta}\, log(1+\frac{x}{\beta})\;\left[1-log\;\alpha\; e^{-a\;\sum_{i=1}^{n}\left\{(1+\frac{x}{\beta})^{\theta}-1\right\}^{b}}\right]. \quad (27)$$

Solving equation (23),(24),(25),(26) and (27) by equating to zero, we can find the maximum likelihood estimates of a,b,α, β and θ.

5. Simulation

In this section a small simulation study to illustrate the efficiency of the ML estimators of APTWL distribution is given. We generate data from the APTWL distribution using the

Table 1: *Simulation study at a=2 , b=2 ,$\alpha = 2$, $\beta = 2$, $\theta = 2$*

n	Parameters	Means	Bias	MSE
	a=2	1.10565	0.89434	0.91377
	b=2	2.39277	-0.39277	0.16138
100	l=2	1.02928	0.97071	0.94228
	p =2	1.01551	1.01550	1.00138
	t=2	1.29873	0.70126	0.49176
	a=2	1.75194	0.24805	0.06152
	b=2	2.38924	-0.38924	0.15151
500	l=2	1.21045	0.78954	0.62337
	p =2	1.27939	0.72060	0.51926
	t=2	1.38891	0.61109	0.47937
	a=2	2.17948	0.17948	0.03221
	b=2	2.03179	-0.03179	0.00101
2000	l=2	1.23165	0.76834	0.61244
	p =2	2.39977	-0.39977	0.15982
	t=2	2.35516	-0.35515	0.12613

Table 2: *Simulation study at a=0.5 , b=0.5 ,$\alpha = 0.5$, $\beta = 0.5$, $\theta = 0.5$*

n	Parameters	Means	Bias	MSE
	a=0.5	0.81558	-0.31558	0.11254
	b=0.5	0.60494	-0.10494	0.03801
100	l=0.5	2.87124	-2.37124	6.35831
	p =0.5	0.13112	0.36887	0.13789
	t=0.5	0.33607	0.16392	0.03643
	a=0.5	0.75076	-0.25076	0.07087
	b=0.5	0.55669	-0.05669	0.00539
500	l=0.5	2.49185	-1.99184	4.46373
	p=0.5	0.14258	0.35741	0.13791
	t=0.5	0.38565	0.11435	0.01722
	a=0.5	0.63824	-0.13824	0.04047
	b=0.5	0.48755	0.01244	0.00071
2000	l=0.5	2.42174	-1.92174	4.08359
	p =0.5	0.24247	0.25752	0.08332
	t=0.5	0.40056	0.09943	0.01081

quantile function given in Eq.(7).The behaviour of the parameters of the APTWL distribution was investigated by conducting simulation studies with the aid of R software. Data sets were generated from the APTWL distribution with a replication number m = 1000; random samples of sizes n = 100, 500 and 2000 were further selected. The simulation was conducted for two different cases using varying true parameter values. The selected true parameter values are $a = 2, b = 2, \alpha = 2, \beta = 2, \theta = 2$ and $a = 0.5, b = 0.5, \alpha = 0.5, \beta = 0.5, \theta = 0.5$ for the first and second cases respectively.

It can be understood from the table 1 and table 2 that mean square error (MSE) reduces for all the selected parameter values as the sample size increases. Also bias reduces as the sample size increases. Hence as sample size increases, the estimates tend towards the true parameter values.

6. Applications

In this study, three lifetime datasets are fitted to demonstrate the adaptability and usefulness of the APTWL distribution. We have fitted the APTWL distribution to the dataset using MLE and compared the proposed APTWL distribution with Weibull Lomax distribution(WL), Exponentiated Lomax distribution(EL), Gamma Lomax distribution(GL), Power Lomax distribution(PL) and Weibull Power Lomax distribution(WPL). The Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) are the goodness of fit statistics that were employed to compare the performances. AIC and BIC are computed as follows:

$$AIC = -2ll + 2k$$
$$BIC = -2ll + k \log(n),$$

where ll is the log likelihood function, k is the parameter number and n is the sample size. We use optim package in R to estimate parameters. Smaller values of the AIC and BIC statistics indicates better model fittings.

6.1. First data set: Strengths of 1.5 cm glass fibres

The glass fibres data set analyzed by Smith and Naylor [18] was used for this comparison. The data set originate from 63 observations of strengths of 1.5cm glass fibres primitively obtained by workers at the UK National Physical Laboratory as reported by Bourguignon *et al.*[2]. The data set is presented below:
0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50, 1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01, 2.24.

The parameter estimates values are shown in Table 3 and the performances of the APTWL distribution with the other competing distributions are shown in Table 4.

Table 3: *MLE for the strengths of 1.5 cm glass fibres*

Distribution	M.L.ESTIMATES				
	α	β	a	b	λ
APTWL	12.7181	2.7532	0.0147	2.1803	4.9414
WL	5.956	6.0375	0.17	3.5281	-
EL	0.0355	74.871	0.0355	-	-
GL	18304.4	20.9077	1312.05	-	-
PL	4.5151	16.196	-	-	77.3271
WPL	62.1361	0.8587	0.019	1.65	63.7374

Table 4: *AIC and BIC measures*

Distribution	-2loglik	AIC	BIC
APTWL	-26.0325	36.03849	46.75417
WL	-30.6798	38.6798	47.2524
EL	-31.95475	69.9095	76.33891
GL	-24.5296	55.05919	61.4886
PL	-32.775	38.6151	39.0218
WPL	-35.1032	36.48121	47.19688

Table 4 compare the APTWL model with the WL, EL, GL, PL and WPL models. We note that the APTWL model gives the lowest values for the AIC and BIC values among all fitted models. So, the APTWL model could be chosen as the best model.

Figure 5 shows the histogram of the data and the estimated pdfs for the fitted models. It is obvious that the APTWL distribution fits the histogram better than the other distributions suggesting that it might be the best model for the given set of data.

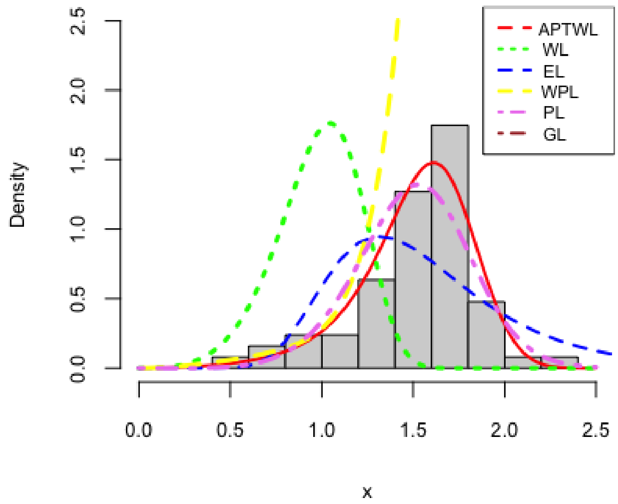

Figure 5: *Estimated pdfs for the first data set*

6.2. Second data set: Remission times of bladder cancer patients

We consider a dataset corresponding to remission times (in months) of a random sample of 128 bladder cancer patients given in Lee and Wang [10]. The observations are as follows:
0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 6.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

For data set 2, the parameter estimates values are shown in Table 5 and further discrepancy criteria for the competing distributions are shown in Table 6.

Table 5: *MLE for the remission times of 128 bladder cancer patients*

Distribution	M.L.ESTIMATES				
	α	β	a	b	λ
APTWL	8.35054	5.78478	87.7104	1.22317	0.04308
WL	0.25661	1.57945	2.42151	1.86389	-
EL	4.5857	24.7414	1.5862	-	-
GL	4.754	20.581	1.5858	-	-
PL	2.07012	1.4276	-	-	34.8626
WPL	62.1361	0.8587	0.0190	1.6500	63.7374

Table 6: *AIC and BIC measures*

Distribution	-2loglik	AIC	BIC
APTWL	-406.8466	817.6932	823.3972
WL	-410.811	829.622	841.03
EL	-407.5037	821.0074	829.5635
GL	-407.5165	821.0331	829.5891
PL	-409.74	825.48	834.036
WPL	-407.0033	824.0066	838.2667

The APTWL model is compared with the WL, EL, GL, PL and WPL models in Table 6. It is evident that out of all the fitted models, the APTWL model provides the lowest values for the AIC and BIC values. Thus the APTWL model may be selected as the optimal model.

The histogram of the data and the estimated pdfs for the fitted models are displayed in Figure 6. It is clear from Figure 6 that the APTWL distribution provides a better fit to the histogram and therefore could be chosen as the best model for the data set.

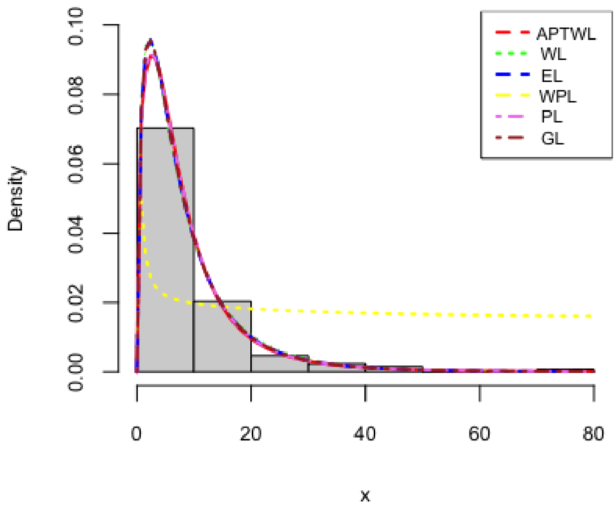

Figure 6: *Estimated pdfs for the second data set*

6.3. Third data set: Breaking stress of carbon fibers

For the third data set, we consider the uncensored data which consist of 100 observations on breaking stress of carbon fibers (in Gba) from Nichols *et al.* [15]. The data are:
3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11,4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53,2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59,2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19,1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69,1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12,1.89, 2.88, 2.82, 2.05, 3.65.

For data set 3, the parameter estimates values are shown in Table 7 and further discrepancy criteria for the competing distributions are shown in Table 8.

Table 7: *MLE for the breaking stress of carbon fibers (in Gba)*

Distribution	M.L.ESTIMATES				
	α	β	a	b	λ
APTWL	4.7392	0.1325	2.1899	5.4454	0.2104
WL	0.15361	2.57945	4.46252	0.56295	-
EL	8.9648	8.2838	14.2225	-	-
GL	1.6499	6.1510	6.9435	-	-
PL	1.6240	3.1692	-	-	29.4556
WPL	0.6134	1.4387	1.2636	2.2953	2.5218

Table 8: *AIC and BIC measures*

Distribution	-2loglik	AIC	BIC
APTWL	“141.1817	292.3634	305.3892
WL	-143.1656	294.7656	306.1296
EL	-146.652	300.7922	308.6077
GL	-143.6743	293.3486	301.1642
PL	-144.0012	296.914	304.7295
WPL	-141.3484	292.6968	305.7226

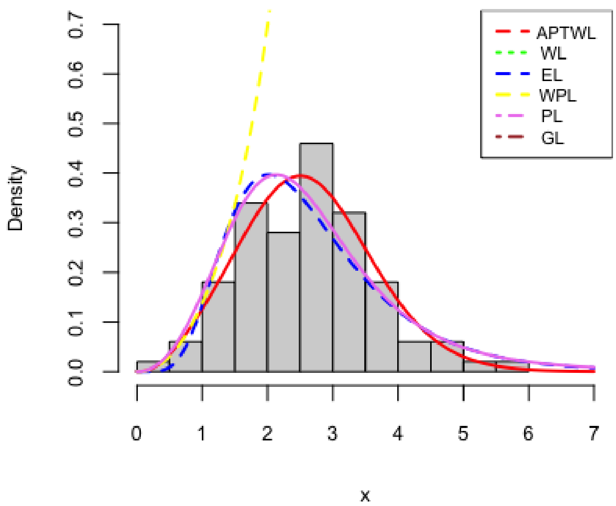

Figure 7: *Estimated pdfs for the third data set*

Table 8 compares the APTWL model with the WL, EL, GL, PL, and WPL models. It is clear that the APTWL model provides the lowest values for the AIC and BIC values among all the fitted models. As a result, the APTWL model might be chosen as the best one.

The histogram of the data and the estimated pdfs for the fitted models are displayed in Figure 7. Figures 7 affirm the results of the analysis that the APTWL distribution is more suitable for the data than the other competing distributions.

7. Conclusion

In this paper we introduced a new model called the Alpha Power Transformed Weibull-Lomax (APTWL) distribution which has bathtub, decreasing, and increasing shapes for the hazard rate function. We derived the structural characteristics of the new distribution including moments, probability weighted moments, generating function and quantile function. Additionally, the Renyi and q entropies are obtained. The APTWL distribution is statistically inferred with the parameters estimated by maximum likelihood estimation. We use three real life datasets for analysing the new model. The new model provides consistently a better fit than the other models namely Weibull Lomax (WL) distribution, Exponentiated Lomax (EL) distribution, Gamma Lomax (GL) distribution, Power Lomax (PL) distribution and Weibull Power Lomax (WPL) distribution. The proposed model will attract wider application in areas such as engineering, survival and lifetime data, economics (income inequality) and others.

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