EXPONENTIATED POISSON-G FAMILY OF DISTRIBUTION: SUB-MODELS, PROPERTIES, ESTIMATION WITH REAL-LIFE APPLICATION

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Abstract

This study proposes a new family of distributions. A study is done on some of its basic characteristics, such as quantile, skewness, kurtosis, hazard rate function, moments, mean deviations, availability and reliability function of successive linear and circular systems, mean time to failure, mean time between failure, and availability, Bonferroni and Lorenz curves, and entropies. Two unique models of the new family are studied in depth once the general class is introduced. The special basis models have been taken from the exponential and Fréchet distributions. The parameters of the model are estimated using maximum likelihood techniques. There is a thorough analysis of percentage points. Three unique real data sets are used to demonstrate the significance of the new family. A comparison is drawn between the suggested distribution family and well-known two-, three-, and four-parameter components. To model actual data, it can be used as an alternative model to various lifetime distributions found in the statistical literature.

Keywords: G-family, Exponentiated model, percentage points, real-life data.

I. Introduction

There are many attempts have been made to introduce new classes in the statistical literature. Several classes have been suggested in the statistical literature by adding one or more factors to construct new distributions. There has been growing interest in introducing and developing more flexible distributions like

Table 1: *Research contributions by various authors*

Contribution	Author(s)	Year of Published
Weibull-G	Bourguignon et. al. [7]	2014
beta Marshall-Olkin family	Alizadeh et. al. [2]	2015
type I half-logistic family	Cordeiro et.al. [9]	2016
The Poisson-G	Hamed & Ibrahim [16]	2017
The generalized transmuted-G	Nofal et.al. [23]	2017
odd flexible Weibull-H	El-Morshedy & Eliwa [13]	2019
odd log-logistic Lindley-G	Alizadeh et. al. [3]	2020
Exponentiated odd Chen-G	Eliwa et.al. [12]	2020
Kumaraswamy Poisson-G	Chakraborty et.al. [8]	2022
Triangle-G	Rahman, H. [25]	2024

The pdf and cdf of the Po-G family of distribution are

$$f_{Po-G}(x,\theta,\xi)=\frac{\theta g(x,\xi)}{e^{\theta}-1}\exp[\theta G(x,\xi)]. \tag{1.1}$$

$$F_{Po-G}(x,\theta,\xi)=\frac{\exp[\theta G(x,\xi)]-1}{e^{\theta}-1}. \tag{1.2}$$

Where $g(x,\xi)$ and $G(x,\xi)$ are the baseline pdf and cdf depending on a parameter vector ξ and $\theta>0$ is a shape parameter. Suppose x is a non-negative random variable that follows an Exponentiated distribution with the following pdf and cdf $f(x)$ and $F(x)$ respectively.

$$g_{\lambda}(x)=\lambda[F(x)]^{\lambda-1}f(x). \tag{1.3}$$

$$G_{\lambda}(x)=[F(x)]^{\lambda}. \tag{1.4}$$

The main goal of this study is to introduce and investigate a generalized family of probability distributions for data modelling with a limited number of parameters but a high degree of flexibility. The remainder of the essay is broken out as follows. We develop a very helpful form for the ExPo-G density function in Section 2. The quantile function (qf), moment generating function (mgf), entropies, and ordinary and incomplete moments are just a few of the general mathematical aspects of the proposed family that are included in Section 3. Section 3 investigates the maximum likelihood estimate of the model parameters. In Section 4, two unique models of this family are provided, along with some plots of their pdfs and hrfs. The results of the proposed models' percentage points are discussed in Section 5. Three applications to actual data sets are made in Section 6 to demonstrate the applicability of two unique models of the proposed family. In Section 7, some last observations are offered.

II. Exponentiated Poisson-G family of Distribution

Let us suppose a random variable $T\in(a,b)$ for $-\infty\leq a<b<\infty$ having a probability density function (pdf) and $W[F(x)]$ be a function of a cumulative distribution function of the random variable X which satisfies some statistical conditions such as $W[F(x)]\in(a,b)$, $W[F(x)]$ is differentiable and monotonically non-decreasing and $W[F(x)]\to a$ as $x\to-\infty$ and $W[F(x)]\to b$ as $x\to\infty$.
Alzaatreh et al. (2013) defined the T-X family cdf by

$$G(x)=\int_{a}^{W[F(x)]}y(t)dt=Y\{W[F(x)]\}.$$

Where $W[F(x)]$ satisfied all the conditions. The corresponding pdf of the T-X family of distribution is

$$g(x)=\left\{\frac{d}{dx}W[F(x)]\right\}y\{W[F(x)]\}.$$

The distribution function of the Expo-G family of distribution is obtained by substituting (1.1) and (1.2) in (1.3) and (1.4). Thus, the pdf and cdf of the ExPo-G family of distribution are presented as

$$f_{ExPo-G}(x,\theta,\lambda,\xi)=\theta\lambda\frac{1}{(e^{\theta}-1)^{\lambda}}g(x,\xi)\exp[\theta G(x,\xi)]\left[\exp\left(\theta G(x,\xi)\right)-1\right]^{\lambda-1}. \tag{2.1}$$

$$F_{ExPo-G}(x,\theta,\lambda,\xi)=\left[\frac{exp\left(\theta G(x,\xi)\right)-1}{e^{\theta}-1}\right]^{\lambda}. \tag{2.2}$$

Where $g(x,\theta,\lambda,\xi)$ and $G(x,\theta,\lambda,\xi)$ are the baseline pdf and cdf depending on a parameter vector ξ and $(\theta,\lambda)>0$ are shape parameters. Let us denote a random variable X having a density function (2.1). The reliability function (rf), hazard rate function (hrf), and cumulative hazard rate function (chrf) of X are, respectively, given by

$$R_{ExPo-G}(x,\theta,\lambda,\xi)=\left[\frac{e^{\theta}-exp\left(\theta G(x,\xi)\right)}{e^{\theta}-1}\right]^{\lambda}. \tag{2.3}$$

$$h_{ExPo-G}(x,\theta,\lambda,\xi)=\frac{\theta\lambda g(x,\xi)\,exp[\theta G(x,\xi)]\left[exp\left(\theta G(x,\xi)\right)-1\right]^{\lambda-1}}{\left(e^{\theta}-exp\left(\theta G(x,\xi)\right)\right)^{\lambda}}. \tag{2.4}$$

$$H_{ExPo-G}(x,\theta,\lambda,\xi)=-\,ln\left[\frac{e^{\theta}-exp\left(\theta G(x,\xi)\right)}{e^{\theta}-1}\right]^{\lambda}. \tag{2.5}$$

III. Statistical Properties

I. Quantile function, Median, Bowley skewness and Moors kurtosis

Let us consider $u \sim U(0,1)$, the u^{th} quantile function of ExPo-G is defined as $Q(u)$ the solution of $G\big(Q(u)\big) = u; Q(u) > 0$. The simplest form can be defined as

$$Q(u) = G^{-1}\left[\frac{1}{\theta} log\left\{1 + u^{\frac{1}{\lambda}}\left(e^{\theta} - 1\right)\right\}\right]. \quad (3.1)$$

where G^{-1} represents the baseline quantile function. The median of the ExPo-G family can be obtained by setting $u = 0.5$. Studying the influence of the shape factors on the skewness and kurtosis using (3.1). The Bowley skewness and Moors kurtosis can be formulated as $B = \frac{Q\left(\frac{3}{4}\right)+Q\left(\frac{1}{4}\right)-2Q\left(\frac{1}{2}\right)}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)}$ and $M = \frac{Q\left(\frac{3}{8}\right)-Q\left(\frac{1}{8}\right)+Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)}$.

II. Ordinary and Incomplete moments, Moment generating Function and Mean Deviation

Let us consider a non-negative random variable$X \sim ExPo - G$, then the r^{th} moment of X is defined as $\mu_r^{'}$and is written as

$$\mu_r^{'} = E(X^r) = \int_0^{\infty} x^r f_{ExPo-G}(x, \theta, \lambda, \xi). \quad (3.2)$$

Using the power series and the generalized binomial expansion, (2.1) can be developed as an infinite mixture of exponential-G (Exp–G) family as

$$f_{ExPo-G}(x, \theta, \lambda, \xi) = \frac{1}{\left(e^{\theta}-1\right)^{\lambda}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\lambda^{j+1}\theta^{k+1}}{(j+1)!(k+1)!} h_{jk+1}(x).$$

$$f_{ExPo-G}(x, \theta, \lambda, \xi) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Omega_{jk}\, h_{jk+1}(x). \quad (3.3)$$

Where $h_{jk+1}(x)$ is the Ex-G family of distribution with power parameter $jk + 1$ and

$$\Omega_{jk} = \frac{\lambda^{j+1}\theta^{k+1}}{\left(e^{\theta}-1\right)^{\lambda}(j+1)!(k+1)!}.$$

The r^{th} moment can be obtained by using (3.3) and is given by

$$\mu_r^{'} = E(X^r) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Omega_{jk}\, E\left({Z^r}_{jk+1}(x)\right). \quad (3.4)$$

$Z_{jk+1}(x)$ is the Ex-G family of distribution with power parameter $jk + 1$.
The mean can be obtained by setting $r = 1$ in (3.4). The i^{th} incomplete moment is defined as $I(x, \theta, \lambda, \xi)$ and is given by

$$I(x, \theta, \lambda, \xi) = \int_{-\infty}^{t} x^i f(x, \theta, \lambda, \xi)\, dx.$$

Using equation (3.3), we get

$$I(x, \theta, \lambda, \xi) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Omega_{jk} \int_{-\infty}^{t} x^i h_{jk+1}(x)\, dx. \quad (3.5)$$

The moment generating function is defined as $M_X(t)$ and given by

$$M_X(t) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Omega_{jk}\, M_{jk+1}(t); \quad M_{jk+1}(t) \text{ is the mgf of } h_{jk+1}(x).$$

For a random variable $X \sim ExPo-G$, the mean deviations about the mean and median can be written as follows

$$\varepsilon_1 = \int_0^{\infty} \left|x - \mu_1^{'}\right| f_{ExPo-G}(x, \theta, \lambda, \xi)dx = 2\mu_1^{'}F\left(\mu_1^{'}\right) - 2I_{(1)}\left(\mu_1^{'}\right) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Omega_{jk}\, H_{jk+1}(x).$$

where $H_{jk+1}(x)$ is the first incomplete moment of Ex-G family.

$$\varepsilon_2 = \int_0^{\infty} |x - Q(0.5)|\, f_{ExPo-G}(x, \theta, \lambda, \xi)dx = \mu_1^{'} - 2I_{(1)}\big(Q(0.5)\big).$$

III. Reliability function for parallel and series systems

Suppose for $n*$ independent components, each component has the ExPo-G family, the reliability of the parallel system (P) is given by

$$R_p(x,\theta,\lambda,\xi) = \left[\frac{e^{\theta} - exp(\theta G(x,\xi))}{e^{\theta}-1}\right]^{\lambda n*}.$$

The reliability of the series system (S) is given by

$$R_s(x,\theta,\lambda,\xi) = \left[\left[\frac{e^{\theta} - exp(\theta G(x,\xi))}{e^{\theta}-1}\right]^{\lambda}\right]^{n*}.$$

IV. Mean time to failure (MTTF), mean time between failure (MTBF) and availability (AvB)

If the MTBF is given as

$$MTBF = \frac{-x}{ln(1 - G(x,\theta_1,\lambda_1,\xi_1))};\ x > 0.$$

If $X \sim ExPo - G(\theta_2,\lambda_2,\xi_2)$ then the MTTF is given as

$$MTTF = E(X) = \mu_1^{'}|(\theta_2,\lambda_2,\xi_2);\ x > 0.$$

The AvB is consider the probability that the component is successful at time x, i.e.

$$AvB = MTTF/MTBF = -\mu_1^{'}|(\theta_2,\lambda_2,\xi_2)\,\frac{ln(1 - G(x,\theta_1,\lambda_1,\xi_1))}{x}.$$

V. Bonferroni and Lorenz curves

Bonferroni and Lorenz curves defined for a given probability π is given by

$$B(\pi) = I_1(q)/\pi\mu_1^{'} \text{ and } L(\pi) = I_1(q)/\mu_1^{'}.$$

Where $q = Q(\pi)$is the quantile function of Xat π.

VI. Entropies

The Rényi entropy of a random variable X represents a measure of variation of the uncertainty. The Rényi entropy is defined by

$$\Upsilon_{\gamma}(x) = \frac{1}{1-\gamma} log \int_{-\infty}^{\infty} g(x)^{\gamma}\, dx;\ 0 < \gamma < 1. \tag{3.6}$$

Using equation (3.3) in (3.6), we get

$$\Upsilon_{\gamma}(x) = \frac{1}{1-\gamma} log\left[\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \Psi_{jk} \int_{-\infty}^{\infty} h_{jk+1}(x)dx\right]; 0 < \gamma < 1.$$

$$\text{Where } \Psi_{jk} = \frac{(\lambda\gamma)^j(\theta\gamma)^k}{(e^{\theta}-1)^{\lambda\gamma}(j+1)!(k+1)!}.$$

The Tsalli's Entropy is denoted by $T_{\gamma}(x)$and given by

$$T_{\gamma}(x) = \frac{1}{1-\gamma} log\left[1 - \left\{\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \Psi_{jk} \int_{-\infty}^{\infty} h_{jk+1}(x)dx\right\}\right]; 0 < \gamma < 1.$$

VII. Parameter Estimation

Here, we determine the Maximum Likelihood Estimation method to estimate the parameters of the new family of distributions from complete samples only. Let $X_1, \ldots, X_n$be a random sample from the ExPo-G family with parameters (θ,λ,ξ). Let $(\theta,\lambda,\xi^T)^T$be the $(p \times 1)$ parameter vector. Then, the log-likelihood function for Θ, say $l = l(\Theta)$, is given by

$$n\, log\, \theta + n\, log\, \lambda - n\lambda\, log(exp(\theta) - 1)^{-(\lambda+1)} + \sum_{i=1}^{n} log\, g\,(x_i;\theta,\lambda,\xi)$$
$$+\theta \sum_{i=1}^{n} G(x_i;\theta,\lambda,\xi) + \left[\theta \sum_{i=1}^{n} G(x_i;\theta,\lambda,\xi) - 1\right]^{\lambda-1}. \tag{3.7}$$

Equation (3.7) can be maximized either directly by using the R (optimum function Ox program (sub-routine MaxBFGS) or by solving the nonlinear likelihood equations obtained by differentiating (3.7).

The score vector components, say $U(\Theta) = \frac{\partial l}{\partial \Theta} = \left(\frac{\partial l}{\partial \theta}, \frac{\partial l}{\partial \lambda}, \frac{\partial l}{\partial \xi}\right)^T = \left(U_\theta, U_\lambda, U_\xi\right)^T$. Setting the nonlinear system of equations $U(\Theta) = 0$and solving them simultaneously yields the MLE $\hat{\Theta} = (\hat{\theta}, \hat{\lambda}, \hat{\xi})$ of $\Theta = (\theta, \lambda, \xi)$. These equations cannot be solved analytically and statistical software can be used to solve them numerically using iterative methods such as the Newton-Raphson type algorithms. For interval estimation of the parameters, we obtain the $p \times p$ observed information matrix $J(\Theta) = \left\{\frac{\partial^3 l}{\partial\theta\partial\lambda\partial\xi}\right\}$, whose elements can be computed numerically. Under standard regularity conditions when $n \to \infty$, the distribution of $\hat{\Theta}$ can be approximated by a multivariate normal $N_p\left(0, J(\hat{\Theta})^{-1}\right)$distribution to obtain confidence intervals for the parameters.

IV. Special ExPo-G models

I. The ExPo-Exponential distribution

Let us consider the pdf and cdf of Exponential distribution with positive parameter α. Then the pdf and cdfof ExPo-Exponential (ExPo-E) distribution is given by

$$f_{ExPo-E}(x, \theta, \lambda, \alpha) = \frac{\alpha\theta\lambda}{(e^{\theta}-1)^{\lambda}} e^{-\alpha x} \, exp[\theta(1 - e^{-\alpha x})] \left[exp\left(\theta(1 - e^{-\alpha x})\right) - 1\right]^{\lambda-1}. \quad (4.1)$$

$$F_{ExPo-E}(x, \theta, \lambda, \alpha) = \left[\frac{exp(\theta(1-e^{-\alpha x})-1}{e^{\theta}-1}\right]^{\lambda}. \quad (4.2)$$

The reliability function is given by

$$R_{ExPo-E}(x, \theta, \lambda, \alpha) = 1 - \left[\frac{exp(\theta(1-e^{-\alpha x})-1}{e^{\theta}-1}\right]^{\lambda}. \quad (4.3)$$

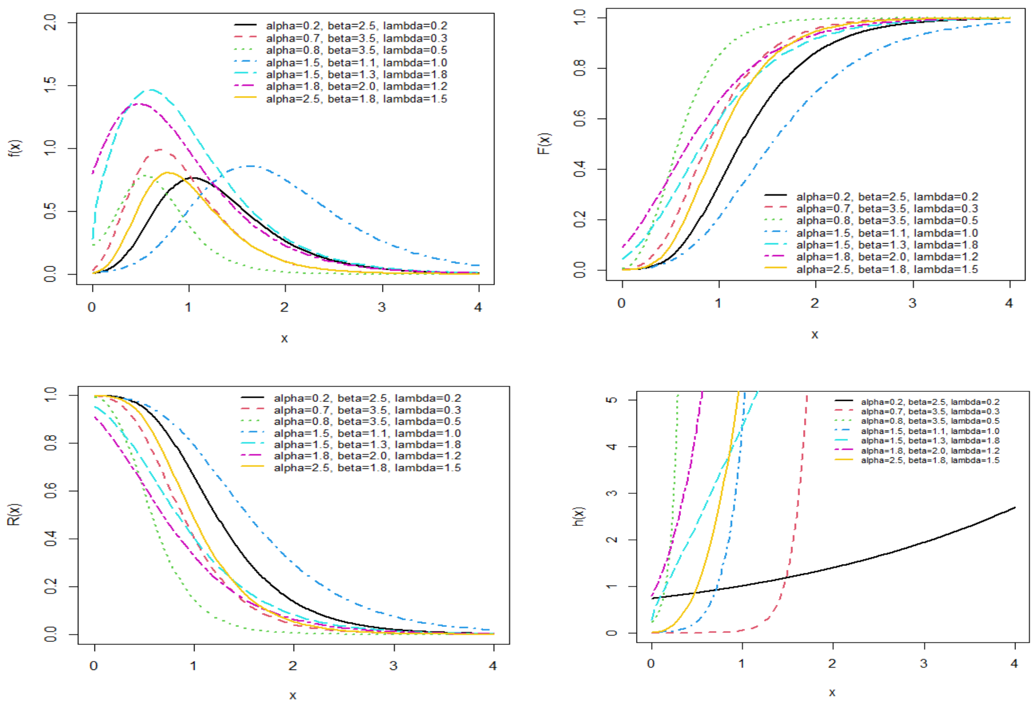

Figure 1: *pdf, cdf, reliability and hrfplots of ExPo-E distribution*

Figure 1 showing the various shapes of the functions with the fluctuation of parameters values. The flexibility of fitting different datasets of the proposed distribution from the increasing or unimodal-bathtub shape of hrf function.

II. The Expo-Frechet (ExPo-Fr) distribution

Let us consider the pdf and cdf of Frechet distribution with positive parameter $\alpha$and β. Then the pdf, cdfand reliability function of ExPo-Frechet (ExPo-Fr) distribution is given by

$$f_{ExPo-Fr}(x,\theta,\lambda,\alpha,\beta) = \frac{\alpha\beta\theta\lambda}{(e^{\theta}-1)^{\lambda}} x^{-(\alpha+1)} e^{-\beta x-\alpha}\, exp\left[\theta\left(e^{-\beta x-\alpha}\right)\right]\left[exp\left(\theta\left(e^{-\beta x-\alpha}\right)\right)-1\right]^{\lambda-1}. \tag{4.4}$$

$$F_{ExPo-Fr}(x,\theta,\lambda,\alpha,\beta) = \left[\frac{exp(\theta(1-e^{-\beta x-\alpha})-1}{e^{\theta}-1}\right]^{\lambda}. \tag{4.5}$$

$$R_{ExPo-Fr}(x,\theta,\lambda,\alpha,\beta) = 1-\left[\frac{exp(\theta(1-e^{-\beta x-\alpha})-1}{e^{\theta}-1}\right]^{\lambda}. \tag{4.6}$$

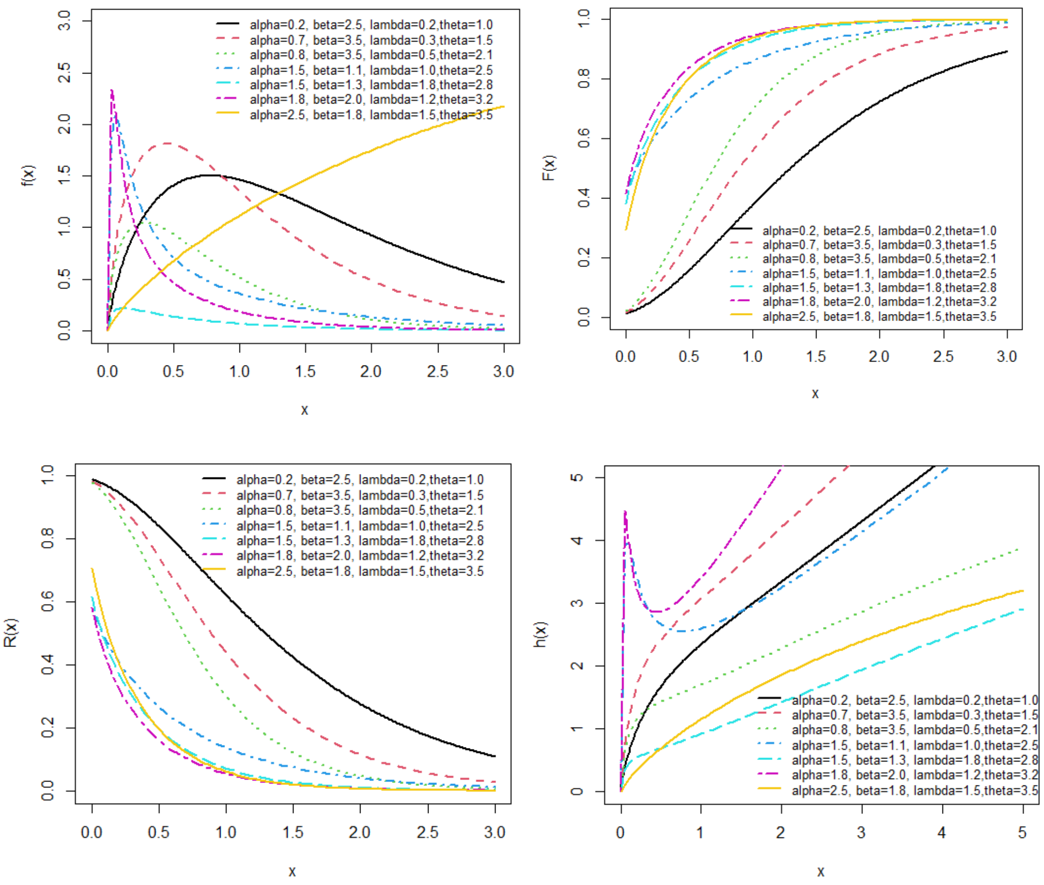

Figure 2: *pdf, cdf, reliability and hrf plots of ExPo-E distribution*

Figure 2 shows the various shapes of the functions with the fluctuation of parameters values. The flexibility of fitting different datasets of the proposed distribution from the increasing shape of hrf function.

V. Percentage Points

A r.v.X is consider a continuous random variable by expecting random values in the interval (a, b) i.e., $a \le x \le b$or more specifically it can assume any value like integral or fraction between certain limits. The number of expecting values are uncertain and infinite and for this reason assigning probability to each number is impossible. Therefore, in a continuous probability distribution we assign probabilities to intervals and not to individual values. For a given probability distribution, the specific value which

a random variable X exceeds with a definite probability is called the percentage point of the distribution.
In this part, a discussion of the percentage points of the proposed models ExPo-E and ExPo-Fr has been attempted. Percentage points of the proposed distributions has been computed at a number of different significance levels for different values of the parameters. The calculations in manual are very complicated, so computer programming R has used to calculate the values.

I. Percentage points of ExPo-E model

Suppose$x_1, x_2, \dots, x_n$are n independent r.v. from ExPo-E with pdf and cdf mentioned in the equation (4.1) and (4.2). The p^{th}percentile equation of ExPo-Eis represented as

$$F_x(x) = P(X \le x) = p.$$
$$x = F^{-1}(p) = Q(p).$$
$$x = -\frac{1}{\alpha} log\left[-\frac{1}{\theta} log\left\{p^{\frac{1}{\lambda}}\left(e^{\theta} - 1\right)\right\}\right]. \quad (5.1)$$

The percent point function of the ExPo-E does not exist in a simple closed form. The numeric computation is not possible in this case. We have used computer programming R to compute the different values for different points. Using the equation (5.1), we compute the percentage points of ExPo-E for $p = 0.01, 0.05, 0.25, 0.50, 0.75, 0.90, 0.95, 0.99$ which has been tabulated in table (2). The parameters are varying different the values to compute the p-table in different cases.

II. Percentage points of ExPo-Fr model

Suppose$x_1, x_2, \dots, x_n$are n independent r.v. from ExPo-Fr with pdf and cdf mentioned in the equation (4.4) and (4.5). The p^{th}percentile equation of ExPo-Fris represented as

$$F_x(x) = P(X \le x) = p.$$
$$x = F^{-1}(p) = Q(p).$$
$$x = -\frac{\alpha}{\beta} log\left[\frac{1}{\theta} log\left\{p^{\frac{1}{\lambda}}\left(e^{\theta} - 1\right)\right\} + 1\right]. \quad (5.2)$$

The percent point function of the ExPo-Fr does not exist in a simple closed form. The numeric computation is not possible in this case. We have used computer programming R to compute the different values for different points. Using the equation (5.2), we compute the percentage points of ExPo-E for $p = 0.01, 0.05, 0.25, 0.50, 0.75, 0.90, 0.95, 0.99$ which has been tabulated in table (2). The parameters are varying different the values to compute the p-table in different cases.

Table 2: *Percentage points of ExPo-E for different values of parameter*

p	0.01	0.05	0.25	0.50	0.75	0.90	0.95	0.99
λ	$\alpha = 3.1, \beta = 2.5$							
0.1	-0.4923	-0.3615	-0.1381	0.0432	0.2311	0.3728	0.4304	0.4825
0.2	-0.2737	-0.1455	0.0704	0.2410	0.4099	0.5293	0.5753	0.6154
0.3	-0.1438	-0.0160	0.1987	0.3674	0.5334	0.6496	0.6940	0.7325
0.4	-0.0490	0.0797	0.2974	0.4704	0.6433	0.7673	0.8155	0.8579
0.5	0.0274	0.1584	0.3827	0.5655	0.7559	0.9007	0.9600	1.0140
β	$\alpha = 0.5, \lambda = 0.2$							
0.1	-11.005	-10.2102	-8.8709	-7.8134	-6.7661	-6.0257	-5.7406	-5.4920
0.2	-9.6189	-8.8239	-7.4846	-6.4271	-5.3798	-4.6394	-4.3543	-4.1057
0.3	-8.8080	-8.0130	-6.6737	-5.6161	-4.5689	-3.8285	-3.5433	-3.2947
0.4	-8.2326	-7.4376	-6.0983	-5.0408	-3.9936	-3.2531	-2.9680	-2.7194
0.5	-7.7863	-6.9913	-5.6520	-4.5945	-3.5473	-2.8068	-2.5217	-2.2731

Table 3: *Percentage points of ExPo-Fr for different values of parameter*

p	**0.01**	**0.05**	**0.25**	**0.50**	**0.75**	**0.90**	**0.95**	**0.99**
θ				$\alpha=1.5, \beta=1, \lambda=3.5$				
1.0	2.2337	0.5666	-0.2034	-0.4426	-0.5667	-0.6193	-0.6346	-0.6461
1.1	1.2455	0.2368	-0.3608	-0.5593	-0.6644	-0.7093	-0.7224	-0.7323
1.2	0.7541	0.0179	-0.4736	-0.6440	-0.7354	-0.7748	-0.7863	-0.7950
1.3	0.4454	-0.1386	-0.5579	-0.7077	-0.7888	-0.8239	-0.8342	-0.8420
1.4	0.2298	-0.2563	-0.6229	-0.7568	-0.8299	-0.8617	-0.8710	-0.8781
1.5	0.0698	-0.3479	-0.6743	-0.7956	-0.8622	-0.8913	-0.8998	-0.9063
$\boldsymbol{\beta}$				$\alpha=0.2, \theta=1.5, \lambda=2.3$				
1.0	0.1399	0.1399	-0.0715	-0.0978	-0.1117	-0.1177	-0.1194	-0.1207
1.1	0.1272	0.1271	-0.0650	-0.0889	-0.1015	-0.1069	-0.1086	-0.1097
1.2	0.1166	0.1166	-0.0596	-0.0815	-0.0931	-0.0981	-0.0995	-0.1006
1.3	0.1076	0.1076	-0.0550	-0.0752	-0.0859	-0.0905	-0.0919	-0.0928
1.4	0.0999	0.0999	-0.0510	-0.0699	-0.0798	-0.0841	-0.0853	-0.0862
1.5	0.0933	0.0933	-0.0476	-0.0652	-0.0744	-0.0785	-0.0796	-0.0804

Percentage points of proposed distributions have been presented in Table 2 and 3. For chosen values of the parameters, different values of have been obtained at different significant levels using R-Programme. From the tables of percentage points of these distributions; it is clear that for fixed values of $p = 0.01, 0.05, 0.25, 0.50, 0.75, 0.90, 0.95, 0.99$ and for fixed positive values of the parameters. From Table2, it is observed that percentage points increase as the parameter λ increases, decreases when the parameters α and β increases; for fixed positive values of other two parameters. From Table 3, it is observed that percentage points decreases when the value of parameter $\alpha, \beta, \theta, \lambda$ increases; for fixed positive values of other parameters.

VI. Data Analysis

This section goes over the empirical significance of using an application to complete real data with the ExPo-G model. The competitive distributions' best-fitting capabilities are determined using specific analytical metrics. To choose the most suited ones, the values of the Akaike Information Criterion (AIC), Hannan-Quinn Information Criterion (HQIC), Corrected Akaike Information Criterion (CAIC), and Bayesian Information Criterion (BIC) were employed. Other goodness-of-fit tests, such as the Cramer-von Mises (W) distance value test, the Kolmogorov-Smirnov (K-S) statistic with accompanying p values, and the loglikelihood function, are also recorded in addition to discriminating tests. The AIC, BIC, CAIC, and HQIC values as well as the W and K-S tests are all lowest for the optimal model. To compare the competitive distributions, the model with the highest p values for the K-S statistics is used. Three data sets have been taken into consideration.

Data Set 1: Bjerkedal (1960) observed and recorded the survival times (in days) of 72 guinea pigs who were infected with virulent tubercle bacilli.

0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Data sets-1 have considered fitting Expo-E with other some models like exponential (Exp), moment exponential (ME), Marshall-Olkin exponential (MO-E), generalized Marshall-Olkin exponential (GMO-E), Kumaraswamy exponential (Kw-E), Beta exponential (BE), Marshall-Olkin Kumaraswamy exponential (MOKw-E) and Kumaraswamy Marshall-Olkin exponential (KwMO-E) distributions and Beta Poisson-Exponential (BP-E).

Data Set 2: Eliwa et al (2022) recorded 101 observations of the fatigue time of 101 6061-T6 aluminium coupons cut parallel to the direction of rolling and oscillates at 18 cycles per second (cps).
5 ,25, 31, 32, 34, 35, 38, 39, 39, 40, 42, 43, 43,43, 44, 44, 47, 47, 48, 49, 49, 49, 51, 54, 55, 55,55, 56, 56, 56, 58, 59, 59, 59, 59, 59, 63, 63, 64,64, 65, 65, 65, 66, 66, 66, 66, 66, 67, 67, 67, 68,69, 69, 69, 69, 71, 71, 72, 73,73, 73, 74, 74, 76,76, 77, 77, 77, 77, 77, 77, 79, 79, 80, 81, 83, 83,84, 86, 86, 87, 90, 91, 92, 92, 92, 92, 93, 94, 97,98, 98, 99, 101, 103, 105, 109, 136, 147.

Data sets 3 have considered fitting Expo-Frdistribution with some competitive models like EoCh Fr, odd Chen Fr (OChFr), Type I generalized exponential Fr (TIGEFr), odd flexible Weibull Fr (OFWFr), Topp-Leaon Fr (ToLeFr), exponentiated Gompertz Fr (EGoFr), exponentiated transmuted Fr (ETrFr), transmuted Fr (TrFr), Gumbel Fr (GuFr), exponentiated Fr (EFr) and Fr.

Table 4: *MLEs and standard errors values for data set 1*

Models	ML Estimator			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$
EXPo-E(α,β,λ)	3.100(2.647)	0.210 (0.023)	0.100(0.002)	-
BP-E($\alpha,\beta,\lambda,\theta$)	3.595 (1.031)	1.482(0.516)	0.014(0.010)	0.724(1.590)
KwMo-E($\alpha,\beta,\lambda,\theta$)	0.373(0.136)	0.299(1.112)	3.478(0.861)	3.306(0.779)
MOKw-E($\alpha,\beta,\lambda,\theta$)	0.008 (0.002)	0.099 (0.048)	2.716 (1.316)	1.986 (0.784)
B-E(α,β,λ)	0.807(0.696)	1.331(0.855)	3.461(1.003)	-
Kw-E(α,β,λ)	3.304(1.106)	1.037(0.764)	1.100(0.614)	-
GMO-E(α,β,λ)	47.635 (44.901)	4.465(1.327)	0.179(0.070)	-
MO-E(α,β)	8.778(3.555)	1.379(0.193)	-	-
ME(α)	0.925 (0.077)	-	-	-
Exp(α)	0.540(0.063)	-	-	-

Table 5: *Log-likelihood, AIC, BIC, CAIC, HQIC, W and KS (p-value) values data set 1*

Models	$-2L$	AIC	$CAIC$	BIC	$HQIC$	$W*$	$K-S$	$p-value$
EXPo-E(α,β,λ)	**141.52**	**147.52**	**147.87**	**154.35**	**151.12**	**0.17**	**0.03**	**1.00**
BP-E($\alpha,\beta,\lambda,\theta$)	199.42	205.42	206.02	214.50	209.02	0.08	0.09	0.81
KwMo-E ($\alpha,\beta,\lambda,\theta$)	201.82	207.82	208.42	216.94	211.42	0.11	0.08	0.73
MOKw-E($\alpha,\beta,\lambda,\theta$)	203.44	209.44	210.04	218.56	213.04	0.12	0.10	0.44
B-E(α,β,λ)	201.38	207.38	207.73	214.22	210.08	0.15	0.11	0.34
Kw-E(α,β,λ)	203.42	209.42	209.77	216.24	212.12	0.11	0.08	0.50
GMO-E(α,β,λ)	204.54	210.54	210.89	217.38	213.24	0.16	0.09	0.51
MO-E(α,β)	204.36	210.36	210.53	214.92	212.16	0.17	0.10	0.43
ME(α)	204.4	210.40	210.45	212.68	211.30	0.25	0.14	0.13
Exp(α)	228.63	234.63	234.68	236.91	235.54	1.25	0.27	0.06

Table 6: *MLEs and confidence intervals values for data set 2*

Models	ML Estimator				
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\gamma}$
EXPo-Fr($\alpha,\beta,\lambda,\theta$)	0.187(0.019)	0.054(0.005)	0.100 (0.214)	0.100(0.025)	-
EOChFr($\alpha,\beta,\lambda,\theta,\gamma$)	0.019	0.257	1.822	4.223	11.500
OChFr($\alpha,\beta,\lambda,\theta$)	49.633	0.629	1.588	444.284	-
TIGEFr($\alpha,\beta,\lambda,\theta$)	16648.994	74.474	5.057	7.531	-
OFWFr($\alpha,\beta,\lambda,\theta$)	9.310	0.380	0.736	312.686	-
ToLeFr($\alpha,\beta,\lambda,\theta$)	35.077	0.783	4.088	59.691	-
EGoFr($\alpha,\beta,\lambda,\theta,\gamma$)	0.013	0.647	1.807	1.158	127.896
TrFr(α,λ,θ)	1.00	-	3.980	136.952	-
GuFr($\alpha,\beta,\lambda,\theta$)	1.968	0.029	0.107	3.457	-
EFr(β,λ,θ)	-	73.221	5.057	51.679	-
Fr(α,β)	5.057	120.782	-	-	-

Table 7: *Log-likelihood, AIC, BIC, CAIC, HQIC, W and KS (p-value) values for the fatigue time of 101 6061-T6 aluminium coupons data set 2*

Models	$-2L$	AIC	$CAIC$	BIC	$HQIC$	$W*$	$K-S$	$p-value$
EXPo-Fr($\alpha,\beta,\lambda,\theta$)	**753.24**	**761.24**	**761.49**	**771.66**	**753.96**	**0.07**	**0.027**	**0.98**
EOChFr($\alpha,\beta,\lambda,\theta,\gamma$)	912.18	920.18	920.43	930.60	912.9	0.08	0.065	0.786
OChFr($\alpha,\beta,\lambda,\theta$)	912.64	920.64	920.89	931.06	913.36	0.07	0.068	0.732
TIGEFr($\alpha,\beta,\lambda,\theta$)	950.38	958.38	958.63	968.80	951.1	0.07	0.133	0.057
OFWFr($\alpha,\beta,\lambda,\theta$)	919.38	927.38	927.63	937.80	920.1	0.15	0.090	0.383
ToLeFr($\alpha,\beta,\lambda,\theta$)	932.70	940.70	940.95	951.12	933.42	0.18	0.121	0.102
EGoFr($\alpha,\beta,\lambda,\theta,\gamma$)	922.60	930.60	930.85	941.02	923.32	0.20	0.107	0.198
TrFr(α,λ,θ)	932.82	940.82	941.07	951.24	933.54	0.20	0.120	0.105
GuFr($\alpha,\beta,\lambda,\theta$)	951.46	959.46	959.71	969.88	952.18	0.25	0.135	0.050
EFr(β,λ,θ)	950.36	958.36	958.61	968.78	951.08	0.27	0.133	0.056
Fr(α,β)	950.36	958.36	958.61	968.78	951.08	0.27	0.133	0.056

Tables 4,5,6 and 7 represent the MLEs with standard errors of the parameters for all the fitted models along with their AIC, BIC, CAIC, HQIC, A, W and KS statistic with p-value from the fitting results of the data sets 4,5 and 6 are presented respectively. The proposed model Expo-E is found to be a better model on the basis of the lowest value different criteria like AIC, BIC, CAIC, HQIC, W and the highest p-value of the KS statistics compared to other introduced models like models Exp, ME, MO-E, GMO-E, Kw-E, BE, MOKw-E, KwMO-E and BP-E considered data set 1.The proposed model Expo-fr is found to be a better model on the basis of the lowest value different criteria like AIC, BIC, CAIC, HQIC, W and the highest p-value of the KS statistics compared to other introduced models like modelsEoCh Fr, odd Chen Fr (OChFr), Type I generalized exponential Fr (TIGEFr), odd flexible Weibull Fr (OFWFr), Topp-Leaon Fr (ToLeFr), exponentiated Gompertz Fr (EGoFr), exponentiated transmuted Fr (ETrFr), transmuted Fr (TrFr), Gumbel Fr (GuFr), exponentiated Fr (EFr) and Fr considered data set 2.

VII. Conclusion

In this article, we propose and study the family of exponentiated Poisson-G distributions (ExPo-G). The main advantage of the ExPo-G family is that practitioners will have a one-parameter class flexible enough to adapt to real data in applied fields. It can be a good alternative to other four parameter families infected with one, two or three parameters. In some real-world circumstances, nevertheless, it might also outperform other kinds of distributions in terms of model fit, albeit this is not always assured. Additionally, a thorough explanation of several of its mathematical features is given. We demonstrate empirically that the ExPo-G family's unique models can offer a better match than other models produced by the aforementioned classes.

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