A COMPARATIVE ANALYSIS OF POWER ISHITA DISTRIBUTION EXPLORING ITS SIGNIFICANT CONTRIBUTION AND PRACTICAL APPLICATIONS

Rashid A. Ganaie[1], Manzoor A. Khanday[2*,] Dilawar A. Bhat[3] and Yudhishther Singh Bagal[4]

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[1]Department of Statistics, Annamalai University, Tamil Nadu-608002, India
[2*]Department of Statistics, Lovely Professional University, Phagwara, Punjab, India
[3]Symbiosis School of Banking and Finance, Symbiosis International University, Pune, India
[4]School of Agriculture, Lovely Professional University, Punjab

rashidau7745@gmail.com, Corresponding author: [2*]manzoorstat@gmail.com,
dilawar.bhat@ssbf.edu.in, ysbagal@gmail.com

Abstract

This research presents the development of a novel statistical model termed the weighted power Ishita distribution. The study delves into its fundamental structural properties, offering a comprehensive theoretical foundation. The parameters of proposed distribution are meticulously estimated through the maximum likelihood estimation method, ensuring robust and reliable results. To validate its applicability and demonstrate its superiority, the model is evaluated using three real-world lifetime data sets, highlighting its efficacy and potential for practical applications.

Keywords: power Ishita distribution, weighted distribution, order statistics, reliability measures

I. Introduction

The fundamental significance of weighted distributions is of great important and plays a major role in real life applications. The weighted distributions provide a standard and flexible approach to deal with efficient modelling of statistical biased data arising from different fields of knowledge even if the standard distributions are not appropriate. The weighted distributions are significant in statistics because it deals with a technique to extend the existing distribution by incorporating an additional parameter. This inclusion of additional parameter brings the existing classical distribution in a more flexible position. The idea of weighted distribution was firstly developed by Fisher [1] to model the ascertainment biases which later on Rao [2] reformulated it in a unified theory for problems where the observations fall in non-experimental, non-replicated and non-random manner. It has been revealed that in some cases weighted distributions does not provide good fit in some datasets because of applied or theoretical point of view. The weighted distributions have wide applications in real life problems, where the probabilities of events are observed and recorded by making some modifications to probabilities of actual occurrence of events considering method of ascertainment. The weighted distributions are used as a tool in selection of appropriate models for observed data

especially if samples are drawn without a proper frame. The weighted distributions are widely applied in diversified fields like medicine, ecology and reliability for the development of proper statistical models. Many researchers developed some weighted probability distributions along with their illustrations from various practical and applied fields [3-14].

Power Ishita distribution is a two parametric distribution developed by Shanker and Shukla [15] which is particular case of Ishita distribution. Its different structural properties have been discussed and its parameters are estimated by using the method of maximum likelihood estimation.

II. Weighted Power Ishita (WPI) Distribution

The probability density function of power Ishita distribution is given by

$$f(x;\theta,\alpha) = \frac{\alpha\,\theta^3}{\left(\theta^3 + 2\right)}\left(\theta + x^{2\alpha}\right)x^{\alpha-1}\,e^{-\theta x^{\alpha}}\;;\; x > 0, \theta > 0,\, \alpha > 0 \tag{1}$$

and cumulative distribution function of power Ishita distribution is given by

$$F(x;\theta,\alpha) = 1 - \left(1 + \frac{\theta x^{\alpha}\left(\theta x^{\alpha} + 2\right)}{\left(\theta^3 + 2\right)}\right)e^{-\theta x^{\alpha}}\;;\; x > 0, \theta > 0, \alpha > 0 \tag{2}$$

Consider random variable X constitutes non-negative condition with probability density function $f(x)$. Let its weight function be $w(x)$ that is non negative then the probability density function of weighted random variable X_w is given by

$$f_w(x) = \frac{w(x)f(x)}{E(w(x))},\;\; x > 0$$

Hence $w(x)$ be the weight function of non - negative condition then $E(w(x)) = \int w(x)f(x)dx < \infty.$

In this paper, we have considered the weight function as $w(x) = x^c$ to get the weighted version of power Ishita distribution and then its probability density function is defined as

$$f_w(x) = \frac{x^c f(x)}{E(x^c)} \tag{3}$$

Here $E(x^c) = \int\limits_0^\infty x^c f(x)dx$

$$E(x^c) = \frac{\left(\theta^3\Gamma\dfrac{(\alpha + c)}{\alpha} + \Gamma\dfrac{(3\alpha + c)}{\alpha}\right)}{\alpha\theta^{\frac{c}{\alpha}}\left(\theta^3 + 2\right)} \tag{4}$$

Therefore, by substituting the equation (1) and (4) in equation (3), we have obtained the required probability density function of weighted power Ishita distribution which is

$$f_w(x) = \frac{\alpha\theta^{\frac{3\alpha + c}{\alpha}}x^{\alpha + c - 1}\left(\theta + x^{2\alpha}\right)e^{-\theta x^{\alpha}}}{\left(\theta^3\,\Gamma\dfrac{(\alpha + c)}{\alpha} + \Gamma\dfrac{(3\alpha + c)}{\alpha}\right)} \tag{5}$$

The cumulative distribution function of weighted power Ishita distribution will be determined by

$$F_w(x) = \int\limits_0^x f_w(x)dx$$

$$= \int_0^x \frac{\alpha \theta^{\frac{3\alpha+c}{\alpha}} x^{\alpha+c-1} \left(\theta + x^{2\alpha}\right) e^{-\theta x^\alpha}}{\left(\theta^3 \Gamma \frac{(\alpha+c)}{\alpha} + \Gamma \frac{(3\alpha+c)}{\alpha}\right)} dx \tag{6}$$

After the simplification of expression (6), we have determined the cumulative distribution function of weighted power Ishita distribution that is

$$F_w(x) = \frac{\left(\theta^3 \gamma\left(\frac{(\alpha+c)}{\alpha}, \theta x^\alpha\right) + \gamma\left(\frac{(3\alpha+c)}{\alpha}, \theta x^\alpha\right)\right)}{\left(\theta^3 \Gamma \frac{(\alpha+c)}{\alpha} + \Gamma \frac{(3\alpha+c)}{\alpha}\right)} \tag{7}$$
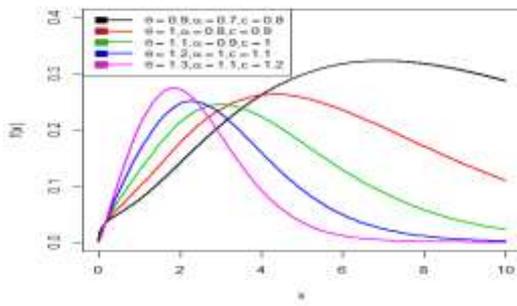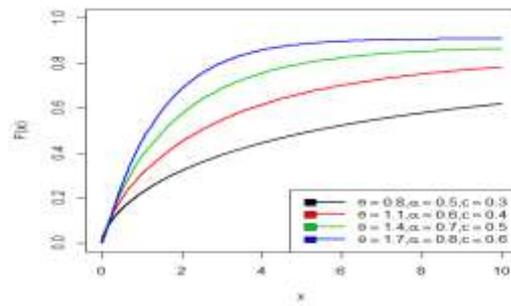

Figure 1: *Pdf plot of WPI distribution*

Figure 2: *Cdf plot of WPI distribution*

III. Reliability measures

In this portion, we have derived the reliability function, hazard function, reverse hazard function and mills ratio of the weighted power Ishita distribution. The reliability function of weighted power Ishita distribution is given by

$$R(x) = 1 - F_w(x)$$

$$R(x) = 1 - \frac{\left(\theta^3 \gamma\left(\frac{(\alpha+c)}{\alpha}, \theta x^\alpha\right) + \gamma\left(\frac{(3\alpha+c)}{\alpha}, \theta x^\alpha\right)\right)}{\left(\theta^3 \Gamma \frac{(\alpha+c)}{\alpha} + \Gamma \frac{(3\alpha+c)}{\alpha}\right)}$$

The hazard function of weighted power Ishita distribution is given by

$$h(x) = \frac{f_w(x)}{1 - F_w(x)}$$

$$h(x) = \frac{x^{\alpha+c-1} \alpha \theta^{\frac{3\alpha+c}{\alpha}} \left(\theta + x^{2\alpha}\right) e^{-\theta x^\alpha}}{\left(\theta^3 \Gamma \frac{(\alpha+c)}{\alpha} + \Gamma \frac{(3\alpha+c)}{\alpha}\right) - \left(\theta^3 \gamma\left(\frac{(\alpha+c)}{\alpha}, \theta x^\alpha\right) + \gamma\left(\frac{(3\alpha+c)}{\alpha}, \theta x^\alpha\right)\right)}$$

The reverse hazard rate function of weighted power Ishita distribution is given by

$$h_r(x) = \frac{f_w(x)}{F_w(x)}$$

$$h_r(x) = \frac{x^{\alpha+c-1} \alpha\theta^{\frac{3\alpha+c}{\alpha}} \left(\theta + x^{2\alpha}\right) e^{-\theta x^{\alpha}}}{\left(\theta^3 \gamma\left(\frac{(\alpha+c)}{\alpha}, \theta x^{\alpha}\right) + \gamma\left(\frac{(3\alpha+c)}{\alpha}, \theta x^{\alpha}\right)\right)}$$

The Mills Ratio of weighted power Ishita distribution is given by

$$M.R = \frac{1}{h_r(x)} = \frac{\left(\theta^3 \gamma\left(\frac{(\alpha+c)}{\alpha}, \theta x^{\alpha}\right) + \gamma\left(\frac{(3\alpha+c)}{\alpha}, \theta x^{\alpha}\right)\right)}{x^{\alpha+c-1} \alpha\theta^{\frac{3\alpha+c}{\alpha}} \left(\theta + x^{2\alpha}\right) e^{-\theta x^{\alpha}}}$$
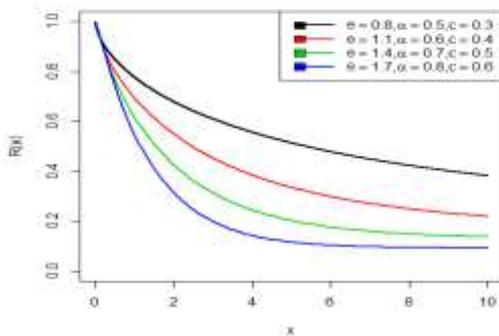

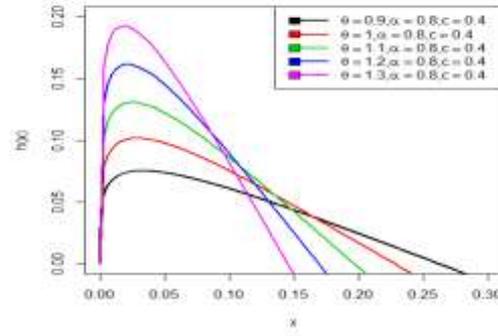**Figure 3:** *Reliability plot of WPI distribution*

Figure 4: *Hazard plot of WPI distribution*

IV. Structural Properties

In this portion, we have derived and described some fundamental statistical properties of weighted power Ishita distribution which are listed as

I. Moments

Consider random variable X represents weighted power Ishita distribution, then rth order moment of weighted power Ishita distribution will be determined as

$$\mu_r' = E(X^r) = \int_0^{\infty} x^r f_w(x) dx$$

$$= \int_0^{\infty} \frac{\alpha\theta^{\frac{3\alpha+c}{\alpha}} x^{\alpha+c+r-1} \left(\theta + x^{2\alpha}\right) e^{-\theta x^{\alpha}}}{\left(\theta^3 \Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(3\alpha+c)}{\alpha}\right)} dx \qquad (8)$$

After simplification of expression (8), we obtain

$$\mu_r' = E(X^r) = \frac{\theta^3 \Gamma\frac{(\alpha+c+r)}{\alpha} + \Gamma\frac{(3\alpha+c+r)}{\alpha}}{\alpha\theta^{\frac{r}{\alpha}}\left(\theta^3 \Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(3\alpha+c)}{\alpha}\right)} \qquad (9)$$

Now substituting $r = 1$ and 2 in equation (9), we obtain the mean and second moment of weighted power Ishita distribution

$$\mu_1' = \frac{\theta^3\Gamma\frac{(\alpha+c+1)}{\alpha} + \Gamma\frac{(3\alpha+c+1)}{\alpha}}{\alpha\theta^{\frac{1}{\alpha}}\left(\theta^3\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(3\alpha+c)}{\alpha}\right)}$$

$$\mu_2' = \frac{\theta^3\Gamma\frac{(\alpha+c+2)}{\alpha} + \Gamma\frac{(3\alpha+c+2)}{\alpha}}{\alpha\theta^{\frac{2}{\alpha}}\left(\theta^3\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(3\alpha+c)}{\alpha}\right)}$$

$$\text{Variance} = \frac{\theta^3\Gamma\frac{(\alpha+c+2)}{\alpha} + \Gamma\frac{(3\alpha+c+2)}{\alpha}}{\alpha\theta^{\frac{2}{\alpha}}\left(\theta^3\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(3\alpha+c)}{\alpha}\right)} - \left(\frac{\theta^3\Gamma\frac{(\alpha+c+1)}{\alpha} + \Gamma\frac{(3\alpha+c+1)}{\alpha}}{\alpha\theta^{\frac{1}{\alpha}}\left(\theta^3\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(3\alpha+c)}{\alpha}\right)}\right)^2$$

$$\text{S.D} = \sqrt{\frac{\theta^3\Gamma\frac{(\alpha+c+2)}{\alpha} + \Gamma\frac{(3\alpha+c+2)}{\alpha}}{\alpha\theta^{\frac{2}{\alpha}}\left(\theta^3\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(3\alpha+c)}{\alpha}\right)} - \left(\frac{\theta^3\Gamma\frac{(\alpha+c+1)}{\alpha} + \Gamma\frac{(3\alpha+c+1)}{\alpha}}{\alpha\theta^{\frac{1}{\alpha}}\left(\theta^3\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(3\alpha+c)}{\alpha}\right)}\right)^2}$$

II. Harmonic mean

The harmonic mean of weighted power Ishita distribution will be determined by using expression

$$\text{H.M} = E\left(\frac{1}{x}\right) = \int_0^\infty \frac{1}{x} f_w(x)dx$$

$$= \int_0^\infty \frac{\alpha\theta^{\frac{3\alpha+c}{\alpha}} x^{\alpha+c-2}\left(\theta + x^{2\alpha}\right)e^{-\theta x^\alpha}}{\left(\theta^3\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(3\alpha+c)}{\alpha}\right)}dx \qquad (10)$$

After simplification of expression (10), we obtain

$$\text{H.M} = \frac{\theta^{\frac{3\alpha+c}{\alpha}}\left(\theta\gamma\left(\frac{(\alpha+c-1)}{\alpha},\theta x^\alpha\right) + \gamma\left(\frac{(3\alpha+c-1)}{\alpha},\theta x^\alpha\right)\right)}{\left(\theta^3\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(3\alpha+c)}{\alpha}\right)}$$

III. Moment generating function and characteristic function

Consider the random variable X consists weighted power Ishita distribution, then MGF of explored distribution will be determined as

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f_w(x)dx$$

After Applying Taylor's series, we get

$$= \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \dots\right) f_w(x)dx$$

$$= \int_0^\infty \sum_{j=0}^\infty \frac{t^j}{j!} x^j f_w(x) dx$$

$$= \sum_{j=0}^\infty \frac{t^j}{j!} \left(\frac{\theta^3 \Gamma \frac{(\alpha+c+j)}{\alpha} + \Gamma \frac{(3\alpha+c+j)}{\alpha}}{\alpha\theta^{\frac{j}{\alpha}} \left(\theta^3 \Gamma \frac{(\alpha+c)}{\alpha} + \Gamma \frac{(3\alpha+c)}{\alpha} \right)} \right)$$

$$M_X(t) = \frac{1}{\alpha \left(\theta^3 \Gamma \frac{(\alpha+c)}{\alpha} + \Gamma \frac{(3\alpha+c)}{\alpha} \right)} \sum_{j=0}^\infty \frac{t^j}{j! \theta^{\frac{j}{\alpha}}} \left(\theta^3 \Gamma \frac{(\alpha+c+j)}{\alpha} + \Gamma \frac{(3\alpha+c+j)}{\alpha} \right)$$

Obviously, the characteristic function of weighted power Ishita distribution is given by

$$M_X(it) = \frac{1}{\alpha \left(\theta^3 \Gamma \frac{(\alpha+c)}{\alpha} + \Gamma \frac{(3\alpha+c)}{\alpha} \right)} \sum_{j=0}^\infty \frac{it^j}{j! \theta^{\frac{j}{\alpha}}} \left(\theta^3 \Gamma \frac{(\alpha+c+j)}{\alpha} + \Gamma \frac{(3\alpha+c+j)}{\alpha} \right)$$

V. Order Statistics

Consider the order statistics $X_{(1)}$, $X_{(2)}$, …, $X_{(n)}$ of a random sample X_1, X_2, …, X_n from a continuous distribution with probability density function $f_x(x)$ and cumulative distribution function $F_x(x)$, then probability density function of rth order statistics $X_{(r)}$ is defined as

$$f_{x(r)}(x) = \frac{n!}{(r-1)!\,(n-r)!} f_X(x)\big(F_X(x)\big)^{r-1} \big(1-F_X(x)\big)^{n-r} \tag{11}$$

Now by applying the equations (5) and (7) in equation (11), we will obtain the probability density function of rth order statistics $X_{(r)}$ of weighted power Ishita distribution which is given by expression

$$f_{x(r)}(x) = \frac{n!}{(r-1)!(n-r)!} \left(\frac{\alpha\theta^{\frac{3\alpha+c}{\alpha}} x^{\alpha+c-1}\big(\theta+x^{2\alpha}\big)e^{-\theta x^\alpha}}{\left(\theta^3 \Gamma \frac{(\alpha+c)}{\alpha} + \Gamma \frac{(3\alpha+c)}{\alpha} \right)} \right) \left(\frac{\left(\theta^3 \gamma\left(\frac{(\alpha+c)}{\alpha}, \theta x^\alpha \right) + \gamma\left(\frac{(3\alpha+c)}{\alpha}, \theta x^\alpha \right) \right)}{\left(\theta^3 \Gamma \frac{(\alpha+c)}{\alpha} + \Gamma \frac{(3\alpha+c)}{\alpha} \right)} \right)^{r-1}$$

$$\times \left(1 - \frac{\left(\theta^3 \gamma\left(\frac{(\alpha+c)}{\alpha}, \theta x^\alpha \right) + \gamma\left(\frac{(3\alpha+c)}{\alpha}, \theta x^\alpha \right) \right)}{\left(\theta^3 \Gamma \frac{(\alpha+c)}{\alpha} + \Gamma \frac{(3\alpha+c)}{\alpha} \right)} \right)^{n-r}$$

Therefore probability density function of higher order statistics $X_{(n)}$ of WPI distribution is given by

$$f_{x(n)}(x) = \frac{n\alpha\theta^{\frac{3\alpha+c}{\alpha}} x^{\alpha+c-1}\big(\theta+x^{2\alpha}\big)e^{-\theta x^\alpha}}{\left(\theta^3 \Gamma \frac{(\alpha+c)}{\alpha} + \Gamma \frac{(3\alpha+c)}{\alpha} \right)} \left(\frac{\left(\theta^3 \gamma\left(\frac{(\alpha+c)}{\alpha}, \theta x^\alpha \right) + \gamma\left(\frac{(3\alpha+c)}{\alpha}, \theta x^\alpha \right) \right)}{\left(\theta^3 \Gamma \frac{(\alpha+c)}{\alpha} + \Gamma \frac{(3\alpha+c)}{\alpha} \right)} \right)^{n-1}$$

and probability density function of first order statistics $X_{(1)}$ of WPI distribution is given by

$$f_{x(1)}(x) = \frac{n\,\alpha\theta^{\frac{3\alpha+c}{\alpha}}\,x^{\alpha+c-1}\left(\theta+x^{2\alpha}\right)e^{-\theta x^{\alpha}}}{\left(\theta^3\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(3\alpha+c)}{\alpha}\right)}\left(1-\frac{\left(\theta^3\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\gamma\left(\frac{(3\alpha+c)}{\alpha},\theta x^{\alpha}\right)\right)}{\left(\theta^3\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(3\alpha+c)}{\alpha}\right)}\right)^{n-1}$$

VI. Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves also termed as income distribution or classical curves which were applied to measure income inequality. The proposed curves are

$$B(p) = \frac{1}{p\mu_1{}'}\int_0^q x\,f_w(x)dx$$

$$\text{and } L(p) = pB(p) = \frac{1}{\mu_1{}'}\int_0^q x\,f_w(x)dx$$

$$\text{Here } \mu_1{}' = \frac{\theta^3\Gamma\frac{(\alpha+c+1)}{\alpha}+\Gamma\frac{(3\alpha+c+1)}{\alpha}}{\alpha\theta^{\frac{1}{\alpha}}\left(\theta^3\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(3\alpha+c)}{\alpha}\right)}$$

$$B(p) = \frac{\alpha\theta^{\frac{1}{\alpha}}\left(\theta^3\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(3\alpha+c)}{\alpha}\right)}{p\left(\theta^3\Gamma\frac{(\alpha+c+1)}{\alpha}+\Gamma\frac{(3\alpha+c+1)}{\alpha}\right)}\int_0^q\frac{\alpha\theta^{\frac{3\alpha+c}{\alpha}}\,x^{\alpha+c}\left(\theta+x^{2\alpha}\right)e^{-\theta x^{\alpha}}}{\left(\theta^3\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(3\alpha+c)}{\alpha}\right)}dx \qquad (12)$$

After simplification of expression (12), we obtain

$$B(p) = \frac{\theta^{\frac{3\alpha+c+1}{\alpha}}}{p\left(\theta^3\Gamma\frac{(\alpha+c+1)}{\alpha}+\Gamma\frac{(3\alpha+c+1)}{\alpha}\right)}\left(\theta\gamma\left(\frac{(\alpha+c+1)}{\alpha},\theta q\right)+\gamma\left(\frac{(3\alpha+c+1)}{\alpha},\theta q\right)\right)$$

$$L(p) = \frac{\theta^{\frac{3\alpha+c+1}{\alpha}}}{\left(\theta^3\Gamma\frac{(\alpha+c+1)}{\alpha}+\Gamma\frac{(3\alpha+c+1)}{\alpha}\right)}\left(\theta\gamma\left(\frac{(\alpha+c+1)}{\alpha},\theta q\right)+\gamma\left(\frac{(3\alpha+c+1)}{\alpha},\theta q\right)\right)$$

VII. Parameter Estimation and Fisher's Information Matrix

In this portion, we have estimated the parameters of weighted power Ishita distribution by using the maximum likelihood estimation. Consider random sample $X_1, X_2,…,X_n$ of size n from weighted power Ishita distribution, then the likelihood function should be stated as

$$L(x) = \prod_{i=1}^{n} f_w(x)$$

$$= \prod_{i=1}^{n} \left(\frac{\alpha \theta^{\frac{3\alpha+c}{\alpha}} x_i^{\alpha+c-1} \left(\theta + x_i^{2\alpha} \right) e^{-\theta x_i^{\alpha}}}{\left(\theta^3 \Gamma \frac{(\alpha+c)}{\alpha} + \Gamma \frac{(3\alpha+c)}{\alpha} \right)} \right)$$

$$= \frac{\alpha \theta^{n\left(\frac{3\alpha+c}{\alpha}\right)}}{\left(\theta^3 \Gamma \frac{(\alpha+c)}{\alpha} + \Gamma \frac{(3\alpha+c)}{\alpha} \right)^n} \prod_{i=1}^{n} \left(x_i^{\alpha+c-1} \left(\theta + x_i^{2\alpha} \right) e^{-\theta x_i^{\alpha}} \right)$$

The log likelihood function is given by

$$\log L = n\left(\frac{3\alpha+c}{\alpha}\right) \log \alpha \theta - n\log\left(\theta^3 \Gamma \frac{(\alpha+c)}{\alpha} + \Gamma \frac{(3\alpha+c)}{\alpha} \right) + (\alpha+c-1)\sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \log\left(\theta + x_i^{2\alpha} \right) - \theta \sum_{i=1}^{n} x_i^{\alpha} \quad (13)$$

Therefore, differentiating the log-likelihood equation (13) with respect to parameters θ and α and c. The following normal equations must be satisfied

$$\frac{\partial \log L}{\partial \theta} = \frac{n\alpha}{\alpha \theta}\left(\frac{3\alpha+c}{\alpha}\right) - n\left(\frac{3\theta^2 \Gamma \frac{(\alpha+c)}{\alpha}}{\left(\theta^3 \Gamma \frac{(\alpha+c)}{\alpha} + \Gamma \frac{(3\alpha+c)}{\alpha} \right)} \right) + \sum_{i=1}^{n} \left(\frac{1}{\left(\theta + x_i^{2\alpha} \right)} \right) - \sum_{i=1}^{n} x_i^{\alpha} = 0$$

$$\frac{\partial \log L}{\partial \alpha} = -\frac{n\theta c}{\alpha^3 \theta} - n\psi\left(\theta^3 \Gamma \frac{(\alpha+c)}{\alpha} + \Gamma \frac{(3\alpha+c)}{\alpha} \right) + \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \left(\frac{x_i^{2\alpha} \log x_i}{\left(\theta + x_i^{2\alpha} \right)} \right) - \theta \sum_{i=1}^{n} x_i^{\alpha} \log x_i = 0$$

$$\frac{\partial \log L}{\partial c} = \frac{n}{\alpha} \log \alpha \theta - n\psi\left(\theta^3 \Gamma \frac{(\alpha+c)}{\alpha} + \Gamma \frac{(3\alpha+c)}{\alpha} \right) + \sum_{i=1}^{n} \log x_i = 0$$

Where $\psi(.)$ is the digamma function.

The above system of non-linear equations are too complicated to solve it algebraically. Therefore, we use R and wolfram mathematics for estimating the parameters of the explored distribution.

In order to apply asymptotic normality results for determining the confidence interval. We have that if $\hat{\gamma} = (\hat{\theta}, \hat{\alpha}, \hat{c})$ denotes the MLE of $\hat{\gamma} = (\hat{\theta}, \hat{\alpha}, \hat{c})$. The results can be stated as

$$\sqrt{n}(\hat{\gamma} - \gamma) \to N_3(0, I^{-1}(\gamma))$$

Where $I(\gamma)$ is the Fisher's Information matrix. i.e.

$$I(\gamma) = -\frac{1}{n} \begin{pmatrix} E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial \theta \partial c}\right) \\ E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \theta}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha \partial c}\right) \\ E\left(\frac{\partial^2 \log L}{\partial c \partial \theta}\right) & E\left(\frac{\partial^2 \log L}{\partial c \partial \alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial c^2}\right) \end{pmatrix}$$

$$E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) = -\frac{n\alpha^2}{(\alpha\theta)^2}\left(\frac{3\alpha+c}{\alpha}\right) - n\left(\frac{\left(\theta^3\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(3\alpha+c)}{\alpha}\right)\left(6\theta\Gamma\frac{(\alpha+c)}{\alpha}\right) - \left(3\theta^2\Gamma\frac{(\alpha+c)}{\alpha}\right)^2}{\left(\theta^3\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(3\alpha+c)}{\alpha}\right)^2}\right) - \sum_{i=1}^{n}\left(\frac{1}{\left(\theta+x_i^{2\alpha}\right)^2}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) = \frac{3n\alpha^2\theta^2 c}{\left(\alpha^3\theta\right)^2} - n\psi'\left(\theta^3\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(3\alpha+c)}{\alpha}\right) + \sum_{i=1}^{n}\left(\frac{\left(\theta+x_i^{2\alpha}\right)x_i^{2\alpha}\left(\log x_i\right)^2 - \left(x_i^{2\alpha}\log x_i\right)^2}{\left(\theta+x_i^{2\alpha}\right)^2}\right) - \theta\sum_{i=1}^{n}\left(x_i^{\alpha}\log x_i\right)^2$$

$$E\left(\frac{\partial^2 \log L}{\partial c^2}\right) = -n\psi'\left(\theta^3\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(3\alpha+c)}{\alpha}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) = -n\psi\left(\frac{3\theta^2\Gamma\frac{(\alpha+c)}{\alpha}}{\theta^3\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(3\alpha+c)}{\alpha}}\right) - \sum_{i=1}^{n}\left(\frac{x_i^{2\alpha}\log x_i}{\left(\theta+x_i^{2\alpha}\right)^2}\right) - \sum_{i=1}^{n}x_i^{\alpha}\log x_i$$

$$E\left(\frac{\partial^2 \log L}{\partial \theta \partial c}\right) = \frac{n\alpha}{\alpha^2\theta} - n\psi\left(\frac{3\theta^2\Gamma\frac{(\alpha+c)}{\alpha}}{\theta^3\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(3\alpha+c)}{\alpha}}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \alpha \partial c}\right) = -\frac{n\theta}{\alpha^3\theta} - n\psi'\left(\theta^3\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(3\alpha+c)}{\alpha}\right)$$

where $\psi(.)'$ is the first order derivative of digamma function. Hence γ is not known, so $I^{-1}(\gamma)$ should be estimated by $I^{-1}(\hat{\gamma})$ and it can be applied to obtain asymptotic confidence intervals for θ, α and c.

VIII. Entropy

The term entropy in statistical terms is clearly defined as increase in the disorganization within a system and it quantifies the diversity, uncertainty or randomness in a system.

I. Renyi entropy

The Renyi entropy is important in quantum information that is used as measure of entanglement and it is also important in economics and statistics as index of diversity

$$e(\beta) = \frac{1}{1-\beta}\log\left(\int f_w^{\beta}(x)dx\right)$$

$$= \frac{1}{1-\beta}\log\int_0^{\infty}\left(\frac{\alpha\theta^{\frac{3\alpha+c}{\alpha}}x^{\alpha+c-1}\left(\theta+x^{2\alpha}\right)e^{-\theta x^{\alpha}}}{\left(\theta^3\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(3\alpha+c)}{\alpha}\right)}\right)^{\beta}dx \qquad (14)$$

After the simplification of expression (14), we obtain

$$e(\beta) = \frac{1}{1-\beta} \log \left(\frac{\theta^{\beta-k}}{\alpha} \left(\frac{\alpha\theta^{\frac{3\alpha+c}{\alpha}}}{\left(\theta^3 \Gamma \frac{(\alpha+c)}{\alpha} + \Gamma \frac{(3\alpha+c)}{\alpha} \right)} \right)^\beta \sum_{k=0}^\infty \binom{\beta}{k} \frac{\Gamma \frac{\beta(\alpha+c-1)+2\alpha k+1}{\alpha}}{(\theta\beta)^{\frac{\beta(\alpha+c-1)+2\alpha k+1}{\alpha}}} \right)$$

Obviously the Tsallis entropy of weighted power Ishita distribution will be determined as

$$S_\tau = \frac{1}{\tau-1} \left(1 - \int_0^\infty f_w{}^\tau(x)dx \right)$$

$$= \frac{1}{\tau-1} \left(1 - \int_0^\infty \left(\frac{\alpha\theta^{\frac{3\alpha+c}{\alpha}} x^{\alpha+c-1} \left(\theta + x^{2\alpha} \right) e^{-\theta x^\alpha}}{\left(\theta^3 \Gamma \frac{(\alpha+c)}{\alpha} + \Gamma \frac{(3\alpha+c)}{\alpha} \right)} \right)^\tau dx \right) \qquad (15)$$

After the simplification of expression (15), we obtain

$$S_\tau = \frac{1}{\tau-1} \left(1 - \frac{\theta^{\tau-j}}{\alpha} \left(\frac{\alpha\theta^{\frac{3\alpha+c}{\alpha}}}{\left(\theta^3 \Gamma \frac{(\alpha+c)}{\alpha} + \Gamma \frac{(3\alpha+c)}{\alpha} \right)} \right)^\tau \sum_{j=0}^\infty \binom{\tau}{j} \frac{\Gamma \frac{\tau(\alpha+c-1)+2\alpha j+1}{\alpha}}{(\tau\theta)^{\frac{\tau(\alpha+c-1)+2\alpha j+1}{\alpha}}} \right)$$

IX. Application

In this portion, we have analysed and examined three real data sets to determine the goodness of fit of weighted power Ishita distribution and then fit has been compared in order to reveal that the weighted power Ishita distribution provides a better fit over power Ishita, Ishita, exponential and Lindley distributions. Analysis of three real lifetime data sets are performed through R software.

Data set 1: The following data set is related to behavioural science collected by Balakrishnan N., et al. [14]. The behavioural and emotional problems of children which can be measured by the scale "General Rating of Affective Symptoms for preschoolers (GRASP)" can be classified with depressive condition or not according to the scale. The study conducted by the researchers in a city at the south Chile, collect the factual data related to scores of GRASP scale of children with frequency in parenthesis are given below
19(6), 20(15), 21(14), 22(9), 23(12), 24(10), 25(6), 26(9), 27(8), 28(5), 29(6), 30(4), 31(3), 32(4), 33, 34, 35(4), 36(2), 37(2), 39, 42, 44

Data set 2: The data set represents the excesses of flood peaks (in m^3/s) Wheat on river near Car cross in Yukon territory, Canada. 72 exceedances of years 1958 to 1984 are recorded, rounded to one decimal place and data set is considered by Choulakian and Stephens [15] and Bodhisuwan et al. [16].
1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 7.0

Data Set 3: The given real lifetime data set represents the survival times (in months) of patients of melanoma studied by Susarla and Vanryzin [17] and the observations of data set are
3.25, 3.50, 4.75, 4.75, 5.00, 5.25, 5.75, 5.75, 6.25, 6.50, 6.50, 6.75, 6.75, 7.78, 8.00, 8.50, 8.50, 9.25, 9.50, 9.50, 10.00, 11.50, 12.50, 13.25, 13.50, 14.25, 14.50, 14.75, 15.00, 16.25, 16.25, 16.50, 17.50, 21.75, 22.50, 24.50, 25.50, 25.75, 27.50, 29.50, 31.00, 32.50, 34.00, 34.50, 35.25, 58.50

To determine the model comparison criterions accompanied with estimating unknown parameters thoroughly, the analysis is performed by applying the technique of R software. In order to compare the performance of weighted power Ishita distribution over power Ishita, Ishita, exponential and Lindley distributions, we consider the criterions like Bayesian Information Criterion *(BIC)*, Akaike Information Criterion *(AIC)*, Akaike Information Criterion Corrected *(AICC)*, Consistent Akaike Information Criterion *(CAIC)*, Shannon entropy *H(X)* and *-2logL*. The distribution is better even if it has smaller criterions of *AIC, BIC, AICC, CAIC, H(X)* and *-2logL* in comparison over other compared distributions. For determining the criterions values given below formulas are used.

$$AIC = 2k - 2\log L, \qquad BIC = k\log n - 2\log L, \qquad AICC = AIC + \frac{2k(k+1)}{n-k-1},$$

$$CAIC = -2\log L + \frac{2kn}{n-k-1} \qquad\qquad H(X) = \frac{-2\log L}{n}$$

Here k is number of parameters, n is sample size and $-2logL$ is maximized value of log-likelihood function under the considered model.

Table 1: *Shows MLE and S.E of data sets of performed distributions*

Data set 1			Data set 2		
Distribution	MLE	S.E	Distribution	MLE	S.E
WPI	$\hat{\alpha} = 0.5335368$ $\hat{\theta} = 0.5652190$ $\hat{c} = 0.0010000$	$\hat{\alpha} = 0.1859930$ $\hat{\theta} = 0.8171725$ $\hat{c} = 1.2418239$	WPI	$\hat{\alpha} = 0.4447812$ $\hat{\theta} = 0.4640847$ $\hat{c} = 0.0010000$	$\hat{\alpha} = 0.1426458$ $\hat{\theta} = 0.8296986$ $\hat{c} = 1.2269077$
PI	$\hat{\alpha} = 0.60967097$ $\hat{\theta} = 0.89794552$	$\hat{\alpha} = 0.07155711$ $\hat{\theta} = 0.11904714$	PI	$\hat{\alpha} = 0.4409878$ $\hat{\theta} = 0.9174580$	$\hat{\alpha} = 0.0283685$ $\hat{\theta} = 0.0656560$
Ishita	$\hat{\theta} = 0.51564530$	$\hat{\theta} = 0.05983518$	Ishita	$\hat{\theta} = 0.256284$	$\hat{\theta} = 0.017182$
Exponential	$\hat{\theta} = 0.17742068$	$\hat{\theta} = 0.03782501$	Exponential	$\hat{\theta} = 0.0838497$	$\hat{\theta} = 0.0098803$
Lindley	$\hat{\theta} = 0.31258813$	$\hat{\theta} = 0.04780664$	Lindley	$\hat{\theta} = 0.156359$	$\hat{\theta} = 0.013089$

Data set 3		
Distribution	MLE	S.E
WPI	$\hat{\alpha} = 0.44939501$ $\hat{\theta} = 0.37266439$ $\hat{c} = 0.00100000$	$\hat{\alpha} = 0.01041128$ $\hat{\theta} = 0.00821445$ $\hat{c} = 0.01003165$
PI	$\hat{\alpha} = 0.45789840$ $\hat{\theta} = 0.75181782$	$\hat{\alpha} = 0.03611135$ $\hat{\theta} = 0.07547508$
Ishita	$\hat{\theta} = 0.19111099$	$\hat{\theta} = 0.01621195$
Exponential	$\hat{\theta} = 0.06386534$	$\hat{\theta} = 0.00941412$
Lindley	$\hat{\theta} = 0.12084543$	$\hat{\theta} = 0.01263491$

Table 2: *Shows comparison, analysis and performance of fitted distributions*

Data sets	Distribution	-2logL	AIC	BIC	AICC	CAIC	H(X)
1	WPI	59.80058	65.80058	69.07371	66.00058	66.00058	0.4822
	PI	98.09093	102.0909	104.273	102.1900	102.1900	0.7910
	Ishita	122.6363	124.6363	125.7273	124.6690	124.6690	0.9890
	Exponential	120.0865	122.0865	123.1776	122.1192	122.1192	0.9684
	Lindley	118.3165	120.3165	121.4075	120.3492	120.3492	0.9541
2	WPI	149.8181	155.8181	162.6481	156.1710	156.1710	2.0808
	PI	330.6732	334.6732	339.2265	334.8471	334.8471	4.5926
	Ishita	597.0425	599.0425	601.3191	599.0996	599.0996	8.2922
	Exponential	500.9405	502.9405	505.2172	502.9976	502.9976	6.9575
	Lindley	524.7423	526.7423	529.019	526.7994	526.7994	7.2880
3	WPI	46.26486	52.26486	57.75078	52.8362	52.8362	1.0057
	PI	194.8344	198.8344	202.4917	199.1134	199.1134	4.2355
	Ishita	334.0115	336.0115	337.8401	336.1024	336.1024	7.2611
	Exponential	345.0919	347.0919	348.9205	347.1828	347.1828	7.5019
	Lindley	333.6992	335.6992	337.5279	335.7901	335.7901	7.2543

It is clearly observed and revealed from the results given above in table 2 that the weighted power Ishita distribution has the smaller *AIC, BIC, AICC, CAIC, H(X)* and *-2logL* values in comparison over power Ishita, Ishita, exponential and Lindley distributions which indicates that the weighted power Ishita distribution leads to better fit as compared over power Ishita, Ishita, exponential and Lindley distributions.

X. Conclusion

In the present study, we have demonstrated a novel class of power Ishita distribution called as weighted power Ishita distribution has been introduced. The developed novel distribution has been executed and generated by using the weighted technique to baseline distribution. i.e. power Ishita distribution. The formulated distribution accomplished with distinguished statistical properties via moments, the mean and variance, shape of behaviour of pdf and cdf, hazard rate function, reliability function, reverse hazard function, moment generating function harmonic mean and order statistics have been determined. Furthermore, its Renyi entropy, Tsallis entropy and Bonferroni and Lorenz curves have been discussed. Additionally, the parameters of presented distribution are estimated by using the technique of maximum likelihood estimation. Finally, the three real world data sets have been analysed and fitted to determine the practicability and superiority of weighted power Ishita distribution. Hence, it is clearly realized from the result that the weighted power Ishita distribution provides a best fit in comparison over power Ishita, Ishita, exponential and Lindley distributions.

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