A PRODUCTION INVENTORY MODEL WITH PROTECTION FOR FEW STAGES OF PRODUCTION

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Abstract

We consider a two server production inventory model with positive service time. Customers arrive to the system according to a Markovian Arrival Process. Service time of customers follow identical but independent phase type distribution. The production of inventory follows (s, S) policy. Production of inventory is by one unit at a time and the production time follows Erlang distribution. While in production shocks occur and consequently breakdown of the production machinery takes place. The shock/damage process occurs according to a Poisson process. After repair, the production process restarts, discarding the item in production. The repair time follows phase type distribution. In order to minimize the product loss due to shock, protection is given to the last k stages of production. Protection of the production process involves additional cost. As a result of this protection, the item, while in the last k stages of production, will not be affected by shocks. Steady state analysis of the model is performed. Some performance measures and distributions of certain important performance characteristics are evaluated. We formulate an optimization problem related to the number of stages of the production process to be protected.

Keywords: production inventory, protection, phase type distribution, Erlang distribution, Markovian Arrival Process

1. Introduction

Inventory with positive service time was first introduced by Melikov and Molchanov[1] and Sigman and Simchi Levi[2], independently of each other. After that a lot of works have been carried out in this area. A survey of inventory with positive service time is given in Krishnamoorthy et al.[3]. Attached to production inventory, positive service time was introduced in Krishnamoorthy et al. [4]. They considered a production inventory system with a single server and server vacation. In that the customer arrival process was assumed to follow a Markovian arrival process(MAP), and the time for producing each item was assumed to have Markovian production scheme. The service process and vacation durations follow independent phase type distributions. Krishnamoorthy et al. in [5] considered two (s, S) production inventory systems with positive service time. Here arrival of customers was according to a Poisson process and service and production time follow independent exponential distributions. In both models, the steady state distributions were obtained in product form. In [6], Baek et al. studied an $M/M/1$ queue with an attached production inventory system. Along with an internal production process following a Poisson process, they considered the (r, Q) inventory control policy. In [6], the customers arriving during stock out period are considered as lost. In [7], Krishnamoorthy et al. introduced the idea of protection in a queueing system where the service process is subject to interruptions. Here the last $m - n$ phases of the Erlang service process were given protection from interruption. Krishnamoorthy et al. in [8] considered (s, S) production inventory system with positive service time and interruptions. Here the customers arrive according to a Poisson process, time for producing each item and service time to each customer follow independent Erlang distributions. Both service process and the production process were subjected to interruptions and certain number of phases in both these processes were given protection from interruption. Anoop and Jacob in [9] considered a queueing system where arrival of customers follows Poisson process. They considered the (s, S) production inventory as servers of the queueing system and the service time follows an exponential distribution. In [10] Yue and Qin considered a production inventory system with positive service times and vacations to the production facility. Customers arrive to the system according to a Poisson process, the service times are exponentially distributed, and has a single production facility that produces one type of product, whose production times are exponentially distributed. Baek et al. in [11] considered an (s, S) production inventory system with an attached Markovian service queue with c servers. In [11], customers leave the system with exactly one item at the service completion epochs and the customers arriving during stock out period are considered as lost. Yue and Qin in [12] considered a production inventory system with service time and product returns, dependent on the characteristics of online shopping behaviors. Here the customers arrive according to a Poisson process, and the service time is exponentially distributed. Jose et al. in [13] considered a single server perishable inventory system in which customers arrive according to a Poisson process. In [13], when a customer arrives if the server is available with a positive level of inventory, that customer enters service. Otherwise, the customer goes to a orbit of infinite capacity with pre determined probability or exits the system with complementary probability. Each customer in the orbit tries to access the server in an exponentially distributed time interval and after every unsuccessful retrial, the customer returns to the orbit with a pre allotted probability or is lost forever with complementary probability.

The highlights of our model are:

- In this model, the production process is subject to shocks which interrupts the production. When the production process is interrupted due to shock, the item in production is lost.
- In order to reduce the effect of loss due to shock, protection is given to the last k stages of production
- The item which is lost due to interruption is sold at scrap value, which is much lower than the selling price of finished goods.
- The cost function is dependent on the number of phases of protection.

In real life, there are many situations in which the production process is subjected to some shock or damage. In such cases, in order to reduce the loss, protection may be given to the production process. For example, suppose the production requires uninterrupted power supply. In case of a power failure, the item in production will be lost. So in order to reduce such loss due to power supply, a protection is provided in the form of a backup power like generator, battery etc. But providing protection requires extra cost. Since economic viability is not feasible for a full time protection, it is provided only in last few stages where major loss is incurred. Our model gains its motivation from such production processes. We encounter many situations in manufacturing which can be modeled as multi server models. Here we consider a two server queueing inventory model, as all the works related considered so far in literature are related to single server models.

The remaining sections of the paper are organized as follows: in Section 2, the model is described. Mathematical formulation of the model is done in Section 3. Steady-state analysis of the model is done in section 4. In section 5 some performance measures are evaluated. In section 6, we derive the expected length of a production period. In section 7, some numerical and graphical illustrations are done. Cost analysis is done in section 8. We conclude the paper in section 9.

2. Model Description

We consider a two server queueing inventory system with production of items to be given to customers at the end of service. Customers arrive to the system according to a Markovian Arrival Process(MAP) with representation (D_0, D_1) of order m. When a customer arrives, if both the servers are idle and there is at least one item in inventory, that customer is served by the first server. When a customer arrives, if only one server is idle and there is at least one item in inventory, that customer is served by the idle server. Otherwise they wait in queue. Backlog is allowed regardless of the inventory/production state. The customers are served one by one on a first come first served(FCFS) basis. The service time of customers by both the servers follow phase type distribution with representation PH$(\boldsymbol{\alpha}, T)$ with m_1 phases, where T^0 is such that $T\boldsymbol{e} + T^0 = \mathbf{0}$. When a customer leaves the system after service completion, the inventory level drops by one unit.

The production of inventory follows (s, S) policy and production is by one unit at a time. The production process is turned on when the inventory level falls to s and it is turned off only when the inventory level reaches S. Production time of each item follows Erlang (β, W) distribution of order m_2 and parameter θ. Here β is the initial probability vector corresponding to this Erlang distribution. Production process is subject to shocks which interrupts the process. Shock/damage process strikes the production machinery only while the production process is on. The arrival of shock process is Poisson with parameter ν. The production machinery fails with the arrival of shock/damage process. When shock/damage occurs, the item being produced is lost. After repair, the production restarts. The repair time of the production machine follows phase type distribution with representation PH(γ, U) with m_3 phases,where U^0 is such that $U\boldsymbol{e} + U^0 = \mathbf{0}$.

When the production process is interrupted due to shock, the item in production is lost and so sold at scrap value, which is lower than the selling price of finished goods. So to reduce the effect of loss due to shock, protection is given to the last k stages of production. This means that if the production process is in the last k stages, protection is provided to the machine against shocks. Thus in the last k stages, it is protected from being interrupted. The first $m_2 - k$ phases of production are unprotected. The protection is given at extra cost. The optimal value of k is investigated for an appropriate cost function.

3. Mathematical Formulation

We introduce the following notations:

- $N(t)$: the number of customers in the system at time t.
- $I(t)$: the number of items in the inventory at time t.
- $P(t)$: the status of production at time t.

$$P(t) = \begin{cases} 0 \text{ production process is off} \\ 1 \text{ production process is on} \\ 2 \text{ production machine is under repair} \end{cases}$$

- $C(t)$: the protection status at time t.

$$C(t) = \begin{cases} 0, \text{ production process is not protected} \\ 1, \text{ production process is protected} \end{cases}$$

- $J_1(t)$: the phase of the service process of server-1 at time t.

- $J_2(t)$: the phase of the service process of server-2 at time t.
- $J_3(t)$: the phase of the production process at time t.
- $R(t)$: the phase of the repair process at time t.
- $A(t)$: the phase of the arrival process at time t.

Then $\{(N(t), I(t), P(t), C(t), J_1(t), J_2(t), J_3(t), R(t), A(t)); t \geq 0\}$ is a continuous time Markov chain on the state space $\Omega = \bigcup_{n=0}^{\infty} l(n)$, where $l(n)$ denotes the collection of states in level n and are defined by considering the first three states as $l(0) = (0, i), l(1) = (1, i)$ and for $n \geq 2$, $l(n) = (n, i)$. Let the ordering of the elements of Ω be lexiographical.

$(0, i) = \{(0, i, 1) \bigcup (0, i, 2)/0 \leq i \leq s\} \bigcup \{(0, i, 0) \bigcup (0, i, 1)) \bigcup (0, i, 2)/s+1 \leq i \leq S-1\} \bigcup (0, S, 0)$.
$(1, i) = (1, 0, 1) \bigcup (1, 0, 2) \bigcup \{(1, i, 1) \bigcup (1, i, 2)/1 \leq i \leq s\}$
$\bigcup \{(1, i, 0) \bigcup (1, i, 1) \bigcup (1, i, 2)/s+1 \leq i \leq S-1\} \bigcup (1, S, 0)$.
$(n, i) = (n, 0, 1) \bigcup (n, 0, 2) \bigcup (n, 1, 1) \bigcup (n, 1, 2) \bigcup \{(n, i, 1) \bigcup (n, i, 2)/2 \leq i \leq s\} \bigcup$
$\{(n, i, 0) \bigcup (n, i, 1) \bigcup (n, i, 2)/s+1 \leq i \leq S-1\} \bigcup (n, S, 0)$.

The transitions, by considering the first three states, are given in the tables 1,2,3,4,5,6 and 7.

The infinitesimal generator Q of the LIQBD (Level Independent Quasi Birth Death) process describing the above two server queueing inventory system is of the form

$$Q = \begin{pmatrix} B_{00} & B_{01} & O & \dots & \dots & \dots & \dots & \dots \\ B_{10} & B_{11} & B_{12} & O & \dots & \dots & \dots & \dots \\ O & B_{21} & A_1 & A_0 & O & \dots & \dots & \dots \\ O & O & A_2 & A_1 & A_0 & O & \dots & \dots \\ O & O & O & A_2 & A_1 & A_0 & O & \dots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

As there are two servers, the behaviour of the system is different for levels 0 and 1. Level 0 corresponds to no customer in the system and level 1 corresponds to 1 customer in the system and hence in service. At level 0, both servers are idle and at level 1, one of the servers is idle. But from level 2 onwards, there are always two or more customers in the system, the two of them are in service. So from level 2 onwards, both servers will be busy, provided there are at least two inventory in the system. So the first three levels are boundary levels and after that the structure gets repeated resulting in a quasi-Toeplitz matrix. The structure of the matrices $B_{00}, B_{01}, B_{10}, B_{11}, B_{12}, B_{21}, A_0, A_1$ and A_2 are given in Appendix A.

4. Steady-State Analysis

For the steady state analysis, we use the Matrix Geometric method by Neuts [14].

4.1. Stability Condition

Theorem-1: The Markov chain with generator Q is stable if and only if

$$\boldsymbol{\pi}[I_{K_4} \otimes D_1]\boldsymbol{e} < \boldsymbol{\pi}_1 H_1 \boldsymbol{e} + \boldsymbol{\pi}_2 H_5 \boldsymbol{e} + \sum_{j=3}^{s} \boldsymbol{\pi}_j H_9 \boldsymbol{e} + \boldsymbol{\pi}_{s+1} H_{10} \boldsymbol{e} + \sum_{j=s+2}^{S-1} \boldsymbol{\pi}_j H_{11} \boldsymbol{e} + \boldsymbol{\pi}_S H_{12} \boldsymbol{e}. \quad (1)$$

Proof. Let $A = A_0 + A_1 + A_2$. Then A is an irreducible matrix and the stationary vector $\boldsymbol{\pi}$ of A is obtained by solving

$$\boldsymbol{\pi} A = 0; \boldsymbol{\pi e} = 1.$$

The Markov chain with generator Q is stable if and only if

$$\boldsymbol{\pi} A_0 e < \boldsymbol{\pi} A_2 \boldsymbol{e},$$

i.e, the system is stable if and only if

$$\boldsymbol{\pi}[I_{K_4} \otimes D_1]\boldsymbol{e} < \boldsymbol{\pi}_1 H_1 \boldsymbol{e} + \boldsymbol{\pi}_2 H_5 \boldsymbol{e} + \sum_{j=3}^{s} \boldsymbol{\pi}_j H_9 \boldsymbol{e} + \boldsymbol{\pi}_{s+1} H_{10} \boldsymbol{e} + \sum_{j=s+2}^{S-1} \boldsymbol{\pi}_j H_{11} \boldsymbol{e} + \boldsymbol{\pi}_S H_{12} \boldsymbol{e}.$$

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4.2. Stationary Distribution

The stationary distribution of the Markov process under consideration is obtained by solving the set of equations

$$\mathbf{x}Q = \mathbf{0}; \mathbf{xe} = \mathbf{1}. \tag{2}$$

Let $\mathbf{x}$ be decomposed in conformity with Q. Then
$\mathbf{x} = (\mathbf{x_0}, \mathbf{x_1}, \mathbf{x_2}, \dots)$, where $\mathbf{x_i} = (\mathbf{x_{i0}}, \mathbf{x_{i1}}, \dots\dots \mathbf{x_{iS}})$, for $i = 0, 1, 2, \dots$.
$\mathbf{x_{ij}} = (\mathbf{x_{ij1}}, \mathbf{x_{ij2}})$, for $j = 0, 1, 2, \dots, s$,
$\mathbf{x_{ij}} = (\mathbf{x_{ij0}}, \mathbf{x_{ij1}}, \mathbf{x_{ij2}})$, for $j = s+1, 2, \dots, S-1$ and
$\mathbf{x_{iS}} = (\mathbf{x_{iS0}})$.
For $j = 0, 1, 2, \dots, S-1$,

$$\mathbf{x_{ij1}} = (\mathbf{x_{ij10}}, \mathbf{x_{ij11}}).$$

From $\mathbf{x}Q = 0$, we get the following equations:

$$\mathbf{x_0} B_{00} + \mathbf{x_1} B_{10} = 0, \tag{3}$$

$$\mathbf{x_0} B_{01} + \mathbf{x_1} B_{11} + \mathbf{x_2} B_{21} = 0, \tag{4}$$

$$\mathbf{x_1} B_{12} + \mathbf{x_2} A_1 + \mathbf{x_3} A_2 = 0, \tag{5}$$

$$\mathbf{x_{i-1}} A_0 + \mathbf{x_i} A_1 + \mathbf{x_{i+1}} A_2 = 0, i = 3, 4, \dots. \tag{6}$$

It may be shown that there exists a constant matrix R such that

$$\mathbf{x_i} = \mathbf{x_{i-1}} R, i = 3, 4, \dots. \tag{7}$$

The sub vectors $\mathbf{x_i}$ are geometrically related by the equation

$$\mathbf{x_i} = \mathbf{x_2} R^{i-1}, i = 3, 4, \dots. \tag{8}$$

R is the minimal non negative solution to the matrix quadratic equation

$$R^2 A_2 + R A_1 + A_0 = O. \tag{9}$$

5. Performance Measures

In this section we evaluate a few performance characteristics of the system that are essential for computation of optimal value of k.

1. Expected number of customers in the system:

$$E[N] = \sum_{i=1}^{\infty} i \mathbf{x_i e}. \tag{10}$$

2. Expected inventory level:
$$E[I] = \sum_{i=0}^{\infty}\sum_{j=1}^{S} j\mathbf{x_{ij}}\mathbf{e}. \tag{11}$$
3. Expected number of customers waiting in the system due to lack of inventory:
$$E[W] = \sum_{i=2}^{\infty}(i-1)\mathbf{x_{i1}}\mathbf{e} + \sum_{i=1}^{\infty} i\mathbf{x_{i0}}\mathbf{e}. \tag{12}$$
4. Probability that both servers are idle:
$$b_0 = \sum_{i=1}^{\infty}\mathbf{x_{i0}}\mathbf{e} + \sum_{\mathbf{j=0}}^{\mathbf{S}}\mathbf{x_{0j}}\mathbf{e}. \tag{13}$$
5. Probability that one of the server is busy and the other idle:
$$b_1 = \sum_{i=1}^{\infty}\mathbf{x_{i1}}\mathbf{e} + \sum_{\mathbf{j=2}}^{\mathbf{S}}\mathbf{x_{1j}}\mathbf{e}. \tag{14}$$
6. Probability that both servers are busy:
$$b_2 = \sum_{i=2}^{\infty}\sum_{j=2}^{S}\mathbf{x_{ij}}\mathbf{e}. \tag{15}$$
7. Probability that production process is off:
$$p_0 = \sum_{i=0}^{\infty}\sum_{j=s+1}^{S}\mathbf{x_{ij0}}\mathbf{e}. \tag{16}$$
8. Probability that production process is on:
$$p_1 = \sum_{i=0}^{\infty}\sum_{j=0}^{S-1}\mathbf{x_{ij1}}\mathbf{e}. \tag{17}$$
9. Probability that production machine is under repair:
$$p_2 = \sum_{i=0}^{\infty}\sum_{j=0}^{S-1}\mathbf{x_{ij2}}\mathbf{e}. \tag{18}$$
10. Probability that production process is on and is protected:
$$p_p = \sum_{i=0}^{\infty}\sum_{j=0}^{S-1}\mathbf{x_{ij11}}\mathbf{e}. \tag{19}$$
11. Probability that production process is on and is not protected:
$$p_u = \sum_{i=0}^{\infty}\sum_{j=0}^{S-1}\mathbf{x_{ij10}}\mathbf{e}. \tag{20}$$
12. Probability that production process is on, is not protected and is in the r^{th} phase of production:
$$p_r = \sum_{i=0}^{\infty}\sum_{j=0}^{S-1}\sum_{s_1=0}^{m_1}\sum_{s_2=0}^{m_1}\mathbf{x_{ij10s_1s_2r}}\mathbf{e}. \tag{21}$$

13. Expected rate of interruptions of the production process in unit time:

$$E[RI] = \nu \sum_{i=0}^{\infty} \sum_{j=0}^{S-1} \mathbf{x_{ij10}e}. \tag{22}$$

14. Probability that inventory level is zero:

$$I_0 = \sum_{i=0}^{\infty} \mathbf{x_{i0}e}. \tag{23}$$

15. The probability that the inventory level is less than or equal to s:

$$p_s = \sum_{i=0}^{\infty} \sum_{j=0}^{s} \mathbf{x_{ij}e}. \tag{24}$$

16. The probability that the maximum storage capacity S is utilized:

$$p_S = \sum_{i=0}^{\infty} \mathbf{x_{iS}e}. \tag{25}$$

6. Expected length of a production period

Here, we derive the expected length of a production period. The expected length of a production period is the time starting from the production switch on time when inventory falls to s, to the time production is switched off for the first time when inventory reaches S. For this, we consider the Markov process
$\{I(t), N(t), P(t), C(t), J_1(t), J_2(t), J_3(t), R(t), A(t); t \geq 0\}$. All variables have the same definition as in section 3, except that $P(t)$ takes the values 1 and 2 only and does not take the value 0. Here we consider the number of inventories as levels and the number of customers in the system is truncated to N. The truncation level N is taken such that, the probability of number of customers in the system going above the truncated size is of the order less than ϵ. Here the inventory level varies from 0 to S and level S is an absorbing state, denoting that production is switched off. The infinitesimal generator of this process is of the form

$$Q_1 = \begin{pmatrix} U_1 & U_1^0 \\ O & O \end{pmatrix}, \tag{26}$$

where U_1 is a sub matrix of Q_1 corresponding to transition among the transient states and U_1^0 corresponding to transition rates to the absorbing state. The structure of U_1 and U_1^0 are as follows:

$$U_1 = \begin{pmatrix} E_{00} & E_{01} & & & & & & \\ E_{10} & E_{11} & E_{12} & & & & & \\ & E_{21} & A_1^{'} & A_0^{'} & & & & \\ & & A_2^{'} & A_1^{'} & A_0^{'} & & & \\ & & & A_2^{'} & A_1^{'} & A_0^{'} & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & & A_2^{'} & A_1^{'} & A_0^{'} \\ & & & & & & A_2^{'} & A_1^{'} \end{pmatrix},$$

and

$$U_1^0 = \begin{pmatrix} O \\ O \\ \vdots \\ A_0^{''} \end{pmatrix}.$$

The structure of the matrices $E_{00}, E_{01}, E_{10}, E_{11}, E_{12}, E_{21}, A_0^{'}, A_1^{'}, A_2^{'}$ and $A_0^{''}$ are given in Appendix B.

The expected period length is the time until absorption of the Markov chain $U_1(t)$. The period length follows PH distributions with representation PH$(\boldsymbol{\alpha_1}, U_1)$, where $\boldsymbol{\alpha_1} = \left(O_{1\times s}, 1, O_{1\times S-s-1}\right)$. Thus we arrive at the following theorem.

Theorem-2: The expected value of the length of the period is approximately given by

$$E[U_1(t)] = -\boldsymbol{\alpha_1} U_1^{-1}\boldsymbol{e}. \tag{27}$$

7. Numerical Examples

In this section, we give some numerical illustrations of variation in performance measures with regard to variation in values of the parameters. Here the MAP describing the arrival is represented by (D_0, D_1). The following values are kept fixed:

$$m = 2, m_1 = 3, m_2 = 5, m_3 = 4, k = 3, s = 4, S = 9, \theta = 20,$$

$$D_0 = \begin{pmatrix} -3.2 & 1 \\ 1.5 & -4.6 \end{pmatrix}; D_1 = \begin{pmatrix} 1.3 & 0.9 \\ 1.4 & 1.7 \end{pmatrix};$$

$$T^0 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}; W^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 20 \end{pmatrix}; U^0 = \begin{pmatrix} 1.8 \\ 2 \\ 2.5 \\ 3 \end{pmatrix};$$

$$\alpha = \begin{pmatrix} 0.2 & 0.4 & 0.4 \end{pmatrix}; T = \begin{pmatrix} -7.5 & 3. & 2.5 \\ 4.8 & -12 & 4.2 \\ 2.6 & 4.5 & -11.1 \end{pmatrix};$$

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix}; W = \begin{pmatrix} -20 & 20 & 0 & 0 & 0 \\ 0 & -20 & 20 & 0 & 0 \\ 0 & 0 & -20 & 20 & 0 \\ 0 & 0 & 0 & -20 & 20 \\ 0 & 0 & 0 & 0 & -20 \end{pmatrix};$$

$$\gamma = \begin{pmatrix} 0.3 & 0.2 & 0.2 & 0.3 \end{pmatrix}; U = \begin{pmatrix} -7.3 & 2 & 1 & 2.5 \\ 2 & -7.3 & 2 & 1.3 \\ 1.6 & 2 & -9.6 & 3.5 \\ 1.2 & 3 & 1.5 & -8.7 \end{pmatrix}.$$

Effect of production interruption rate ν on various performance measures

Tables 8 and 9 shows the variation in various measures of performance for different values of ν

- From table 8 and figures 1 and 2 it is clear that the expected number of customers in the system and the expected number of customers waiting in the system due to lack of inventory increases as ν increases. But the expected inventory level decreases as ν increases. This is because as the production interruption rate ν increases, the production gets interrupted more frequently. This reduces the production rate. As a result, the inventory level decreases and so the expected number of customers in the system and the expected number of customers waiting in the system due to lack of inventory increases.
- As the production interruption rate ν increases, we see from table 8 and figure 3 that probability that both servers are idle decreases slightly, probability that one of the server is busy and the other idle increases and probability that both servers are busy decreases.

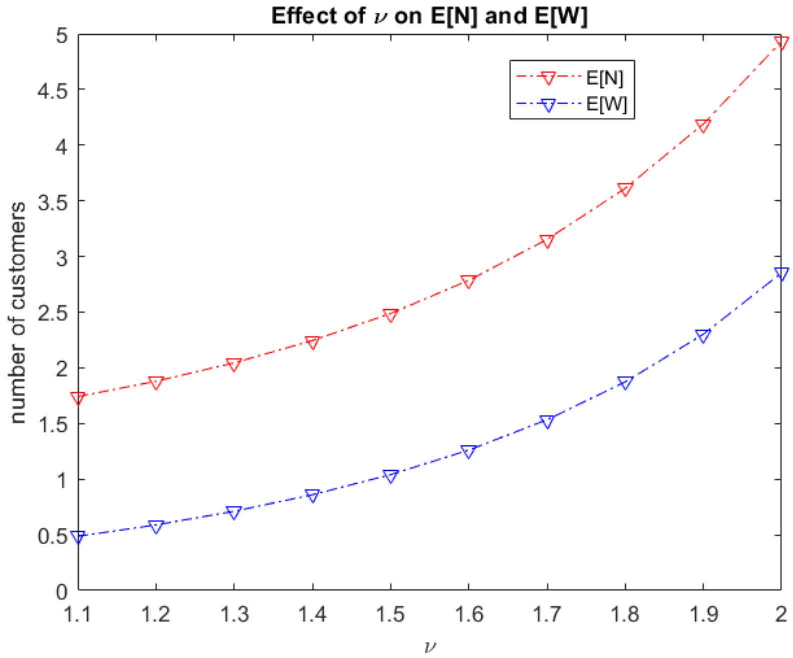

Figure 1

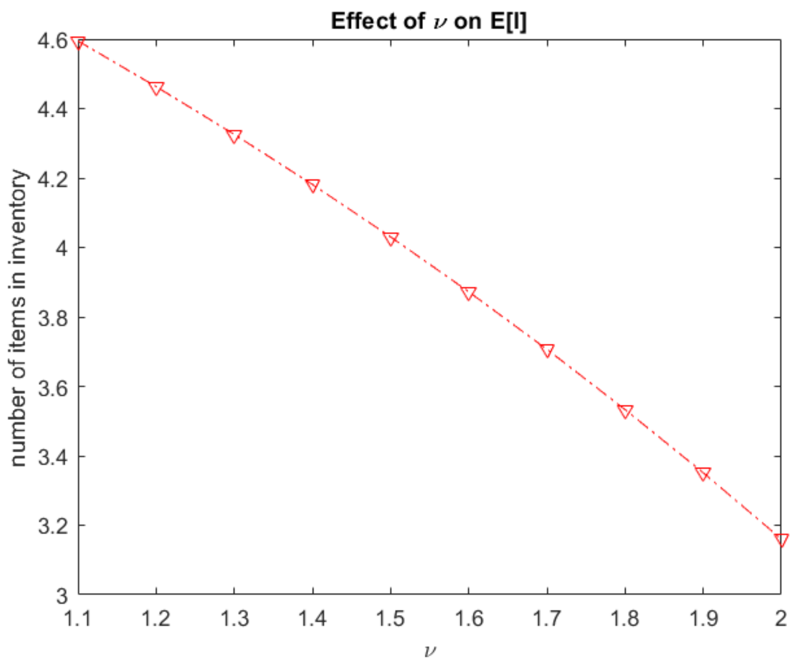

Figure 2

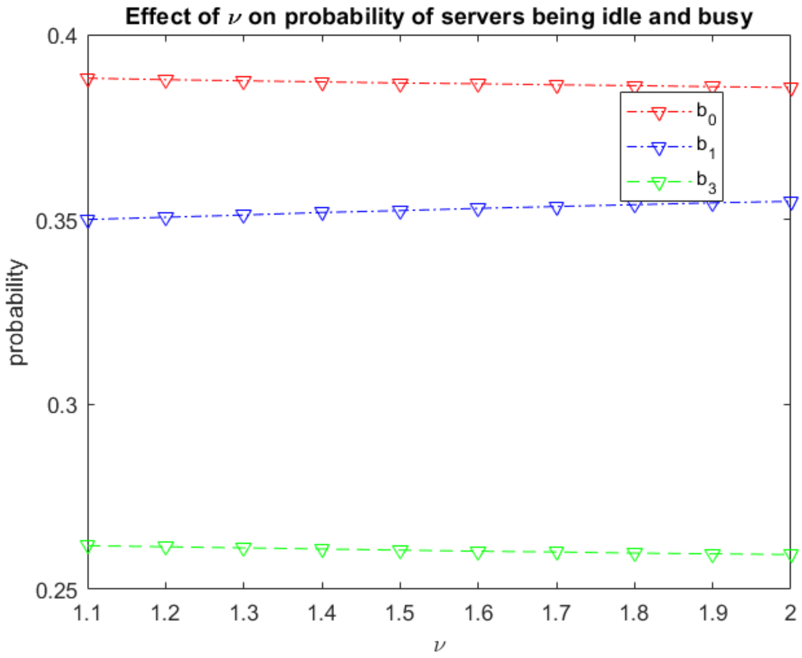

Figure 3

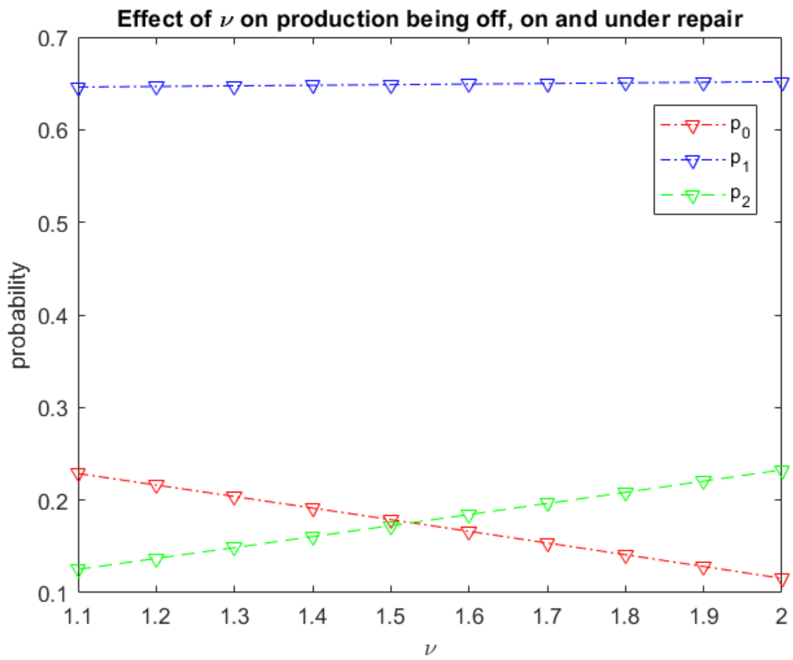

Figure 4

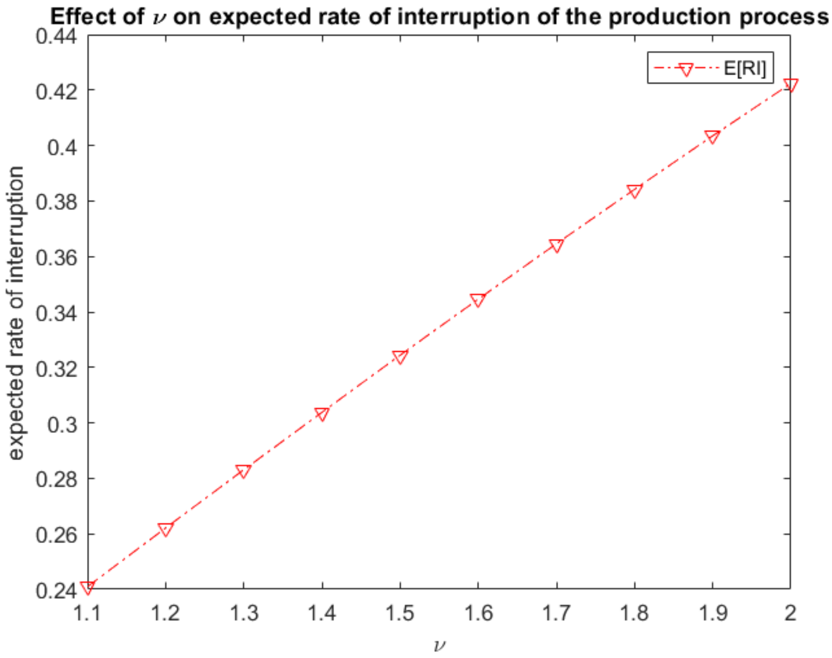

Figure 5

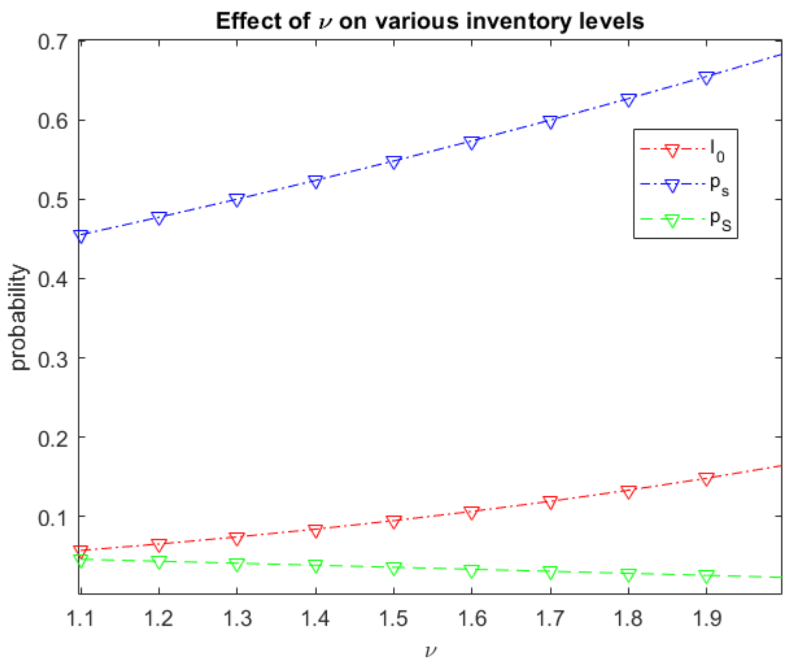

Figure 6

This is because as ν increases, the inventory level decreases and both servers are busy only if there is at least two inventory. So the probability that both servers are busy decreases slightly as ν increases. Thus probability that one of the server is busy and the other idle increases slightly as it happens if there is at least one inventory.

- It is clear from table 8 and figure 4 that as ν increases, the probability that production is off decreases and the probability that production is on and under repair increases. This is due to the fact that as the production interruption rate ν increases, the production gets interrupted more frequently and the probability that the production process is under repair increases. This reduces the inventory level and so the probability that production is on increases and the probability that production is off decreases.
- Also it is clear from table 9 and figure 5 that as the production interruption rate ν increases, the expected rate of interruption also increases.
- From table 9 and figure 6, we see that as ν increases, the probability that the inventory level is 0 increases, the probability that the inventory level is less than or equal to s increases and the probability that the maximum storage capacity S is used decreases. This is because as ν increases, the inventory level decreases.

8. Cost Analysis

For determining the optimal number of production phases to be protected, we construct a cost function. For the cost analysis we define the following costs:

- C_1: Holding cost per customer per unit time.
- C_2: Holding cost per item per unit time.
- C_3: Unit time cost for running the production machinery.
- C_4: Unit time cost incurred due to protection of the production machinery.
- C_5: Unit time cost incurred due to repair of the production machinery.
- C_6: Fixed cost for a production period.
- a: Cost incurred if the item is lost at the first phase of production.
- b: Cost incurred if the item is lost after the first phase of production.

The expected total cost is

$$ETC = C_1 * E[N] + C_2 * E[I] + C_3 * p_1 + C_4 * p_p + C_5 * p_2 + C_6 / E[X(t)] + \sum_{l=1}^{m_2-k} \nu * a * b^{l-1} * p_l.$$

We fix the following values:
$\nu = 1, C_1 = 100, C_2 = 110, C_3 = 240, C_4 = 300, C_5 = 250, C_6 = 450, a = 50, b = 2.$
The values of $s, S, m, m_1, m_2, m_3, D_0, D_1, T, T^0, W, W^0, U, U^0, \alpha, \beta, \gamma$ are same as in section 7.

- From table 10 and figure 7, we see that the optimal value of the cost function is at $k = 3$. Thus it is optimal to give protection to the last 3 phases of production. The total cost is maximum when $k = 1$. This means that if we give protection only to the last phase, the total cost is high.

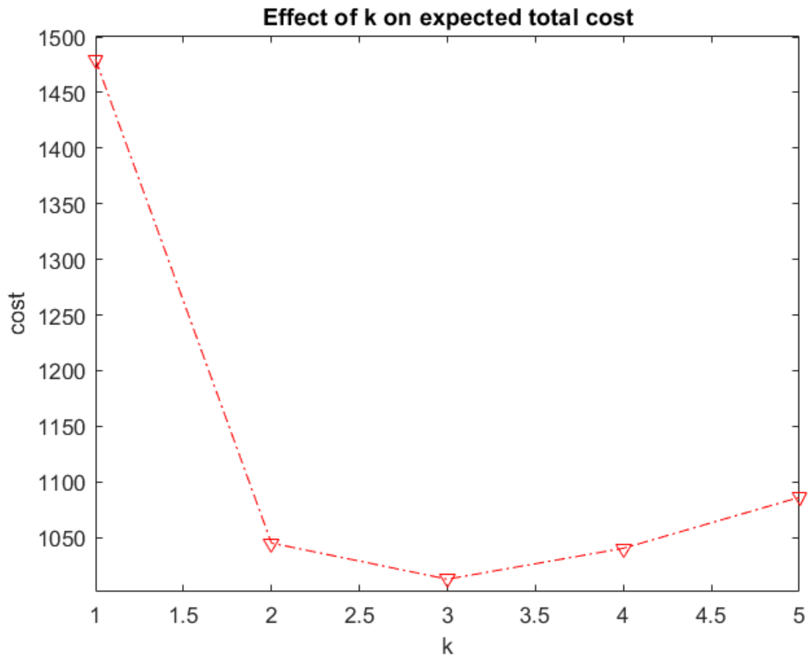

Figure 7

9. Conclusion

In this paper, we considered a production inventory model with two servers and positive service time. Production process is subject to shocks which interrupts the process and so to reduce the effect of loss due to shock, protection is given to the last k stages of production. Steady state analysis of the model is performed and some performance measures are evaluated. The expected length of a production period is calculated. We investigated numerically the variation in performance measures with regard to variation in values of the parameters. We formulated an optimization problem related to the number of stages of the production process to be protected.

Appendix A

Let $\beta_1 = \beta \otimes I_m$, $V_0 = \begin{pmatrix} \nu \otimes I_{m_2-k} & O \\ O & O \end{pmatrix}$, β is a $1 \times m_2$ matrix with one in the first column and zero elsewhere.

$$W^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \theta \end{pmatrix};\ W = \begin{pmatrix} -\theta & \theta & 0 & 0 & 0 \\ 0 & -\theta & \theta & 0 & 0 \\ 0 & 0 & -\theta & \theta & 0 \\ 0 & 0 & 0 & -\theta & \theta \\ 0 & 0 & 0 & 0 & -\theta \end{pmatrix};$$

$V = \begin{pmatrix} \nu \\ \vdots \\ \nu \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, V is a column matrix with first $m_2 - k$ rows entries as ν and zero for the last k rows.

$$B_{01} = \begin{pmatrix} I_{(m_1+m_2)} \otimes D_1 & O & O & O \\ O & I_s \otimes E_3 & O & O \\ O & O & I_{(S-s-1)} \otimes E_4 & O \\ O & O & O & F_3 \end{pmatrix}, \tag{28}$$

$$B_{00} = E_1 + E_2, \tag{29}$$

$$B_{11} = E_8 + E_9, \tag{30}$$

$$B_{10} = \begin{pmatrix} O & O & O & O & O \\ I_s \otimes H_1 & O & O & O & O \\ O & H_2 & O & O & O \\ O & O & I_{(S-s-2)} \otimes H_3 & O & O \\ O & O & O & H_4 & O \end{pmatrix}, \tag{31}$$

$$B_{12} = \begin{pmatrix} I_{(2m_1+1)(m_2+m_3)} \otimes D_1 & O & O & O \\ O & I_{(s-1)} \otimes E_6 & O & O \\ O & O & I_{(S-s-1)} \otimes E_7 & O \\ O & O & O & F_6 \end{pmatrix}, \tag{32}$$

$$B_{21} = \begin{pmatrix} O & O & O & O & O & O \\ H_1 & O & O & O & O & O \\ O & I_{s-1} \otimes H_5 & O & O & O & O \\ O & O & H_6 & O & O & O \\ O & O & O & I_{(S-s-2)} \otimes H_7 & O & O \\ O & O & O & O & H_8 & O \end{pmatrix}, \tag{33}$$

$$A_2 = \begin{pmatrix} O & O & O & O & O & O & O \\ H_1 & O & O & O & O & O & O \\ O & H_5 & O & O & O & O & O \\ O & O & I_{s-2} \otimes H_9 & O & O & O & O \\ O & O & O & H_{10} & O & O & O \\ O & O & O & O & I_{S-s-2} \otimes H_{11} & O & O \\ O & O & O & O & O & H_{12} & O \end{pmatrix}, \tag{34}$$

$$A_0 = I_{K_4} \otimes D_1, \tag{35}$$

$$A_1 = A_{11} + A_{12}, \tag{36}$$

$E_1 = \begin{pmatrix} I_{s+1} \otimes C_1 & O & O \\ O & I_{S-s-1} \otimes C_4 & O \\ O & O & D_0 \end{pmatrix}$,

$E_2 = \begin{pmatrix} O & I_s \otimes C_2 & O & O & O \\ O & O & C_3 & O & O \\ O & O & O & I_{S-s-2} \otimes C_5 & O \\ O & O & O & O & C_6 \\ O & O & O & O & O \end{pmatrix}$,

$E_3 = \begin{pmatrix} F_1 & O \\ O & F_2 \end{pmatrix}$, $E_4 = \begin{pmatrix} F_3 & O & O \\ O & F_1 & O \\ O & O & F_2 \end{pmatrix}$,

$$H_1 = \begin{pmatrix} h_{11} & O \\ O & h_{12} \end{pmatrix}, H_2 = \begin{pmatrix} h_{21} & O \\ h_{11} & O \\ O & h_{12} \end{pmatrix},$$

$$H_3 = \begin{pmatrix} h_{31} & O & O \\ O & h_{11} & O \\ O & O & h_{12} \end{pmatrix}, H_4 = \begin{pmatrix} h_{31} & O \end{pmatrix},$$

$$E_8 = \begin{pmatrix} C_1 & O & O & O \\ O & I_s \otimes C_8 & O & O \\ O & O & I_{S-s-1} \otimes C_{11} & O \\ O & O & O & C_{14} \end{pmatrix},$$

$$E_6 = \begin{pmatrix} F_4 & O \\ O & F_5 \end{pmatrix}, E_7 = \begin{pmatrix} F_6 & O & O \\ O & F_4 & O \\ O & O & F_5 \end{pmatrix},$$

$$E_9 = \begin{pmatrix} O & C_7 & O & O & O & O \\ O & O & I_{s-1} \otimes C_9 & O & O & O \\ O & O & O & C_{10} & O & O \\ O & O & O & O & I_{S-s-2} \otimes C_{12} & O \\ O & O & O & O & O & C_{13} \\ O & O & O & O & O & O \end{pmatrix},$$

$$H_5 = \begin{pmatrix} h_{51} & O \\ O & h_{52} \end{pmatrix}, H_6 = \begin{pmatrix} h_{61} & O \\ h_{51} & O \\ O & h_{52} \end{pmatrix},$$

$$H_7 = \begin{pmatrix} h_{71} & O & O \\ O & h_{51} & O \\ O & O & h_{52} \end{pmatrix}, H_8 = \begin{pmatrix} h_{71} & O \end{pmatrix},$$

$$K_4 = (m_2 + m_3)[1 + 2m_1 + (S-2)m_1^2] + (S-s)m_1^2,$$

$$A_{11} = \begin{pmatrix} C_1 & O & O & O & O \\ O & C_8 & O & O & O \\ O & O & I_{s-1} \otimes G_2 & O & O \\ O & O & O & I_{S-s-1} \otimes G_3 & O \\ O & O & O & O & G_4 \end{pmatrix},$$

$$A_{12} = \begin{pmatrix} O & C_7 & O & O & O & O & O \\ O & O & G_1 & O & O & O & O \\ O & O & O & I_{s-2} \otimes G_5 & O & O & O \\ O & O & O & O & G_6 & O & O \\ O & O & O & O & O & I_{S-s-2} \otimes G_7 & O \\ O & O & O & O & O & O & G_8 \\ O & O & O & O & O & O & O \end{pmatrix},$$

$$H_9 = \begin{pmatrix} h_{91} & O \\ O & h_{92} \end{pmatrix}, H_{10} = \begin{pmatrix} h_{10} & O \\ h_{91} & O \\ O & h_{92} \end{pmatrix},$$

$$H_{11} = \begin{pmatrix} h & O & O \\ O & h_{91} & O \\ O & O & h_{92} \end{pmatrix}, H_{12} = \begin{pmatrix} h & O & O \end{pmatrix},$$

$$F_1 = \begin{pmatrix} O & \alpha \otimes (I_{m_2} \otimes D_1) \end{pmatrix}, F_2 = \begin{pmatrix} O & \alpha \otimes (I_{m_3} \otimes D_1) \end{pmatrix},$$

$$F_3 = \begin{pmatrix} O & \alpha \otimes D_1 \end{pmatrix}, F_4 = \begin{pmatrix} \alpha \otimes (I_{m_1 m_2} \otimes D_1) \\ I_{m_1} \otimes (\alpha \otimes I_{m_2}) \otimes D_1 \end{pmatrix},$$

$$F_5 = \begin{pmatrix} \alpha \otimes (I_{m_1 m_3} \otimes D_1) \\ I_{m_1} \otimes (\alpha \otimes I_{m_2}) \otimes D_1 \end{pmatrix}, F_6 = \begin{pmatrix} \alpha \otimes (I_{m_1} \otimes D_1) \\ I_{m_1} \otimes (\alpha \otimes D_1) \end{pmatrix},$$

$$C_1 = \begin{pmatrix} W \oplus D_0 - V_0 & (\gamma \otimes V) \otimes I_m \\ U^0 \otimes \beta_1 & U \oplus D_0 \end{pmatrix},$$

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$C_2 = \begin{pmatrix} W^0 \otimes \beta_1 & O \\ O & O \end{pmatrix}$, $C_3 = \begin{pmatrix} O & W^0 \otimes \beta_1 & O \\ O & O & O \end{pmatrix}$,
$C_4 = \begin{pmatrix} D_0 & O \\ O & C_1 \end{pmatrix}$, $C_5 = \begin{pmatrix} O \\ C_3 \end{pmatrix}$,
$C_6 = \begin{pmatrix} O \\ W^0 \otimes I_m \\ C_3 \end{pmatrix}$, $C_7 = \begin{pmatrix} \begin{pmatrix} 0 & (\alpha \otimes W^0) \otimes \beta_1 \end{pmatrix} & O \\ O & O \end{pmatrix}$,
$C_8 = \begin{pmatrix} I_2 \otimes [T \oplus (W \oplus D_0)] - I_{2m_1} \otimes V_0 & I_{2m_1} \otimes [(\gamma \otimes V) \otimes I_m] \\ I_{2m_1} \otimes (U^0 \otimes \beta_1) & I_2 \otimes [T \oplus (U \oplus D_0)] \end{pmatrix}$,
$C_9 = \begin{pmatrix} I_{2m_1} \otimes (W^0 \otimes \beta_1) & O \\ O & O \end{pmatrix}$,
$C_{10} = \begin{pmatrix} O & I_{2m_1} \otimes (W^0 \otimes \beta_1) & O \\ O & O & O \end{pmatrix}$,
$C_{11} = \begin{pmatrix} c_{11} & c_{12} \end{pmatrix}$,
$c_{11} = \begin{pmatrix} I_2 \otimes [T \oplus D_0] & O \\ O & I_2 \otimes [T \oplus (W \oplus D_0)] - I_{2m_1} \otimes V_0 \\ O & I_{2m_1} \otimes (U^0 \otimes \beta_1) \end{pmatrix}$,
$c_{12} = \begin{pmatrix} O \\ I_{2m_1} \otimes (\gamma \otimes V) \otimes I_m \\ I_2 \otimes [T \oplus (U \oplus D_0)] \end{pmatrix}$,
$C_{12} = \begin{pmatrix} O \\ C_{10} \end{pmatrix}$, $C_{13} = \begin{pmatrix} O \\ I_{2m_1} \otimes W^0 \otimes I_m \\ O \end{pmatrix}$,
$C_{14} = I_2 \otimes [T \oplus D_0]$, $G_1 = \begin{pmatrix} g_1 & O \\ O & O \end{pmatrix}$,
$G_2 = \begin{pmatrix} g_{21} & I_{m_1^2} \otimes ((\gamma \otimes V) \otimes I_m) \\ I_{m_1^2} \otimes (U^0 \otimes \beta_1) & g_{22} \end{pmatrix}$,
$G_3 = \begin{pmatrix} g_{31} & O & O \\ O & g_{21} & I_{m_1^2} \otimes ((\gamma \otimes V) \otimes I_m) \\ O & I_{m_1^2} \otimes (U^0 \otimes \beta_1) & g_{22} \end{pmatrix}$,
$G_4 = T \oplus [T \oplus D_0]$, $G_5 = \begin{pmatrix} I_{m_1^2} \otimes (W^0 \otimes \beta_1) & O \\ O & O \end{pmatrix}$,
$G_6 = \begin{pmatrix} O & I_{m_1^2} \otimes (W^0 \otimes \beta_1) & O \\ O & O & O \end{pmatrix}$,
$G_7 = \begin{pmatrix} O & O & O \\ O & I_{m_1^2} \otimes (W^0 \otimes \beta_1) & O \\ O & O & O \end{pmatrix}$,
$G_8 = \begin{pmatrix} O \\ I_{m_1^2} \otimes (W^0 \otimes I_m) \\ O \end{pmatrix}$,
$g_1 = \begin{pmatrix} (\alpha \otimes (I_{m_1} \otimes W^0)) \otimes \beta_1 \\ I_{m_1} \otimes ((\alpha \otimes W^0) \otimes \beta_1) \end{pmatrix}$,
$g_{21} = T \oplus [T \oplus (W \oplus D_0)] - I_{m_1^2 m} \otimes V_0$,
$g_{22} = T \oplus [T \oplus (U \oplus D_0)]$,
$g_{31} = T \oplus [T \oplus D_0]$,
$h_{11} = \begin{pmatrix} T^0 \otimes I_{m_2 m} \\ T^0 \otimes I_{m_2 m} \end{pmatrix}$, $h_{12} = \begin{pmatrix} T^0 \otimes I_{m_3 m} \\ T^0 \otimes I_{m_3 m} \end{pmatrix}$,
$h_{21} = \begin{pmatrix} T^0 \otimes \beta I_m \\ T^0 \otimes \beta I_m \end{pmatrix}$, $h_{31} = \begin{pmatrix} T^0 \otimes I_m \\ T^0 \otimes I_m \end{pmatrix}$,

$$h_{51} = \begin{pmatrix} T^0 \otimes I_{m_1 m_2 m} & I_{m_1} \otimes (T^0 \otimes I_{m_2 m}) \end{pmatrix},$$
$$h_{52} = \begin{pmatrix} T^0 \otimes I_{m_1 m_3 m} & I_{m_1} \otimes (T^0 \otimes I_{m_3 m}) \end{pmatrix},$$
$$h_{61} = \begin{pmatrix} (T^0 \otimes I_{m_1}) \otimes \beta_1 & I_{m_1} \otimes (T^0 \otimes \beta_1) \end{pmatrix},$$
$$h_{71} = \begin{pmatrix} T^0 \otimes I_{m m_1} & I_{m_1} \otimes (T^0 \otimes I_m \end{pmatrix},$$
$$h_{91} = (T^0 \otimes \alpha) \otimes I_{m_1 m_2 m} + I_{m_1} \otimes ((T^0 \otimes \alpha) \otimes I_{m_2 m}),$$
$$h_{92} = (T^0 \otimes \alpha) \otimes I_{m_1 m_3 m} + I_{m_1} \otimes ((T^0 \otimes \alpha) \otimes I_{m_3 m}),$$
$$h_{10} = ((T^0 \otimes \alpha) \otimes I_{m_1}) \otimes \beta_1 + I_{m_1} \otimes ((T^0 \otimes \alpha) \otimes \beta_1),$$
$$h = (T^0 \otimes \alpha) \otimes I_{m_1 m} + I_{m_1} \otimes ((T^0 \otimes \alpha) \otimes I_m).$$

Appendix B

$$E_{00} = \begin{pmatrix} C_1 & C_1' & & & \\ & C_1 & C_1' & & \\ & & \ddots & \ddots & \\ & & & C_1 & C_1' \\ & & & & C_1'' \end{pmatrix},$$

$$E_{01} = \begin{pmatrix} C_2 & O \\ O & I_N \otimes C_7 \end{pmatrix}, \; E_{10} = \begin{pmatrix} O & O \\ I_N \otimes H_1 & O \end{pmatrix},$$

$$E_{11} = \begin{pmatrix} C_1 & E_3 & & & & \\ & C_8 & E_3' & & & \\ & & C_8 & E_3' & & \\ & & & \ddots & \ddots & \\ & & & & C_8 & E_3' \\ & & & & & C_8' \end{pmatrix},$$

$$E_{12} = \begin{pmatrix} C_2 & O & O \\ O & C_9 & O \\ O & O & I_{N-1} \otimes G_1 \end{pmatrix}, \; E_{21} = \begin{pmatrix} O & O & O \\ H_1 & O & O \\ O & I_{N-1} \otimes H_5 & O \end{pmatrix},$$

$$A_1' = \begin{pmatrix} C_1 & E_3 & & & & \\ & C_8 & E_6 & & & \\ & & G_2 & E_6' & & \\ & & & \ddots & \ddots & \\ & & & & G_2 & E_6' \\ & & & & & G_2' \end{pmatrix},$$

$$A_2' = \begin{pmatrix} O & O & O & O \\ H_1 & O & O & O \\ O & H_5 & O & O \\ O & O & I_{N-2} \otimes H_9 & O \end{pmatrix},$$

$$A_0' = \begin{pmatrix} C_2 & O & O \\ O & C_9 & O \\ O & O & I_{N-1} \otimes G_5 \end{pmatrix}, \; A_0'' = \begin{pmatrix} C_6' & O & O \\ O & C_{13}' & O \\ O & O & I_{N-1} \otimes G_8' \end{pmatrix},$$

$$C_1' = I_{m_2+m_3} \otimes D_1,$$
$$C_1'' = \begin{pmatrix} W \oplus (D_0 + D_1) - V_0 & (\gamma \otimes V) \otimes I_m \\ U^0 \otimes \beta_1 & U \oplus (D_0 + D_1) \end{pmatrix},$$
$$C_8' = \begin{pmatrix} I_2 \otimes [T \oplus (W \oplus (D_0 + D_1))] - I_{2m_1} \otimes V_0 & I_{2m_1} \otimes [(\gamma \otimes V) \otimes I_m] \\ I_{2m_1} \otimes (U^0 \otimes \beta_1) & I_2 \otimes [T \oplus (U \oplus (D_0 + D_1))] \end{pmatrix},$$
$$E_3 = \begin{pmatrix} e_{31} & O \\ O & e_{32} \end{pmatrix}, \; E_6 = \begin{pmatrix} e_{61} & O \\ O & e_{62} \end{pmatrix},$$

$$E_3^{'} = I_{2m_1(m_2+m_3)} \otimes D_1,\ E_6^{'} = I_{m_1^2(m_2+m_3)} \otimes D_1,$$

$$G_2^{'} = \begin{pmatrix} g'_{21} & I_{m_1^2} \otimes [(\gamma \otimes V) \otimes I_m] \\ I_{m_1^2} \otimes (U^0 \otimes \beta_1) & g'_{22} \end{pmatrix},$$

$$C_6^{'} = \begin{pmatrix} W^0 \otimes I_m \\ O \end{pmatrix},\ C_{13}^{'} = \begin{pmatrix} I_{2m_1} \otimes (W^0 \otimes I_m) \\ O \end{pmatrix},$$

$$G_8^{'} = \begin{pmatrix} I_{m_1^2} \otimes (W^0 \otimes I_m) \\ O \end{pmatrix},$$

$$e_{31} = \begin{pmatrix} O & \alpha \otimes (I_{m_2} \otimes D_1) \end{pmatrix},\ e_{32} = \begin{pmatrix} O & \alpha \otimes (I_{m_3} \otimes D_1) \end{pmatrix},$$

$$e_{61} = \begin{pmatrix} \alpha \otimes (I_{m_1 m_2} \otimes D_1) \\ I_{m_1} \otimes (\alpha \otimes (I_{m_2} \otimes D_1)) \end{pmatrix},\ e_{62} = \begin{pmatrix} \alpha \otimes (I_{m_1 m_3} \otimes D_1) \\ I_{m_1} \otimes (\alpha \otimes (I_{m_3} \otimes D_1)) \end{pmatrix},$$

$$g_{21}^{'} = T \oplus [T \oplus (W \oplus (D_0 + D_1))] - I_{m_1^2} \otimes V_0,$$

$$g_{22}^{'} = T \oplus [T \oplus (U \oplus (D_0 + D_1))].$$

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References

[1] A. Z. Melikov and A. A. Molchanov. "Stock optimization in transportation/storage systems". In: *Cybernetics and Systems Analysis* 28.3 (1992), pp. 484–487. ISSN: 1573-8337. DOI: 10.1007/bf01125431.

[2] Karl Sigman and David Simchi-Levi. "Light traffic heuristic for anM/G/1 queue with limited inventory". In: *Annals of Operations Research* 40.1 (Dec. 1992), pp. 371–380. ISSN: 1572-9338. DOI: 10.1007/bf02060488.

[3] Achyutha Krishnamoorthy, Dhanya Shajin, and Viswanath C. Narayanan. "Inventory with Positive Service Time: a Survey". In: *Queueing Theory 2*. Wiley, Apr. 2021, pp. 201–237. ISBN: 9781119755234. DOI: 10.1002/9781119755234.ch6.

[4] A. Krishnamoorthy and Viswanath C. Narayanan. "Production inventory with service time and vacation to the server". In: *IMA Journal of Management Mathematics* 22.1 (2011), pp. 33–45. DOI: 10.1093/imaman/dpp025.

[5] A. Krishnamoorthy and Narayanan C. Viswanath. "Stochastic decomposition in production inventory with service time". In: *European Journal of Operational Research* 228.2 (2013), pp. 358–366. ISSN: 0377-2217. DOI: 10.1016/j.ejor.2013.01.041.

[6] Jung Woo Baek and Seung Ki Moon. "The M/M/1 queue with a production-inventory system and lost sales". In: *Applied Mathematics and Computation* 233 (May 2014), pp. 534–544. ISSN: 0096-3003. DOI: 10.1016/j.amc.2014.02.033.

[7] A. Krishnamoorthy, B. Gopakumar, and C.N. Viswanath. "An M/E^m/1 Queue With Protected and Unprotected Phases From Interruption". 5th International Conference on Queueing Theory and Network Applications. 2010.

[8] A. Krishnamoorthy, Sajeev S. Nair, and Viswanath C. Narayanan. "Production inventory with service time and interruptions". In: *International Journal of Systems Science* 46.10 (Oct. 2013), pp. 1800–1816. ISSN: 1464-5319. DOI: 10.1080/00207721.2013.837538.

[9] Anoop N. Nair and M. J. Jacob. "An (s,S) Production Inventory Controlled Self-Service Queuing System". In: *Journal of Probability and Statistics* 2015 (2015), pp. 1–8. ISSN: 1687-9538. DOI: 10.1155/2015/505082.

[10] Dequan Yue and Yaling Qin. "A Production Inventory System with Service Time and Production Vacations". In: *Journal of Systems Science and Systems Engineering* 28.2 (Feb. 2019), pp. 168–180. ISSN: 1861-9576. DOI: 10.1007/s11518-018-5402-8.

[11] Jung Woo Baek and Seung Ki Moon. "A production–inventory system with a Markovian service queue and lost sales". In: *Journal of the Korean Statistical Society* 45.1 (Mar. 2016), pp. 14–24. ISSN: 1226-3192. DOI: 10.1016/j.jkss.2015.05.002.

[12] Yaling Qin and Dequan Yue. "Performance Analyses of Production Inventory Systems Considering Service Time and Product Returns of Online Shopping". In: *Journal of Systems Science and Complexity* 32.3 (2018), pp. 888–906. ISSN: 1559-7067. DOI: 10.1007/s11424-018-7230-9.

[13] K. P. Jose and P. S. Reshmi. "A production inventory model with deteriorating items and retrial demands". In: *OPSEARCH* 58.1 (Aug. 2020), pp. 71–82. ISSN: 0975-0320. DOI: 10.1007/s12597-020-00471-8.

[14] M. F. Neuts. *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach*. Baltimore: Johns Hopkins University Press, 1981. ISBN: 978-0-486-68342-3.

Tables

Table 1: *Transition table corresponding to arrivals*

From	To	Rate	Description
$(n,0,1)$	$(n+1,0,1)$	$I_{m_2} \otimes D_1$	arrival when both servers are idle and no inventory in the system, production is on, $n \geq 0$
$(n,0,2)$	$(n+1,0,2)$	$I_{m_3} \otimes D_1$	arrival when both servers are idle and no inventory in the system, production process is under repair, $n \geq 0$
$(0,i,1)$	$(1,i,1)$	$\begin{pmatrix} 0 & \alpha \otimes (I_{m_2} \otimes D_1) \end{pmatrix}$	$1 \leq i \leq S-1$, arrival when both servers are idle, at least one inventory in the system, and production is on
$(0,i,2)$	$(1,i,2)$	$\begin{pmatrix} 0 & \alpha \otimes (I_{m_3} \otimes D_1) \end{pmatrix}$	$1 \leq i \leq S-1$, arrival when both servers are idle, at least one inventory in the system, and production process is under repair
$(0,i,0)$	$(1,i,0)$	$\begin{pmatrix} 0 & \alpha \otimes D_1 \end{pmatrix}$	$s+1 \leq i \leq S$, arrival when both servers are idle and production is off
$(n,1,1)$	$(n+1,1,1)$	$I_{2m_1m_2} \otimes D_1$	arrival when one server is busy, the other server is idle, only one inventory in system, production is on, $n \geq 1$
$(n,1,2)$	$(n+1,1,2)$	$I_{2m_1m_3} \otimes D_1$	arrival when one server is busy, the other server is idle, only one inventory in system, production process is under repair, $n \geq 1$

Table 2: *Transition table corresponding to arrivals*

From	To	Rate	Description
$(1,i,1)$	$(2,i,1)$	$\begin{pmatrix} \alpha \otimes (I_{m_1 m_2} \otimes D_1) \\ I_{m_1} \otimes (\alpha \otimes (I_{m_2} \otimes D_1)) \end{pmatrix}$	$2 \le i \le S-1$, arrival when one server is busy, the other server is idle, at least two inventory in the system, and production is on
$(1,i,2)$	$(2,i,2)$	$\begin{pmatrix} \alpha \otimes (I_{m_1 m_3} \otimes D_1) \\ I_{m_1} \otimes (\alpha \otimes (I_{m_3} \otimes D_1)) \end{pmatrix}$	$2 \le i \le S-1$, arrival when one server is busy, the other server is idle, at least two inventory in the system, and production process is under repair
$(1,i,0)$	$(2,i,0)$	$\begin{pmatrix} \alpha \otimes (I_{m_1} \otimes D_1) \\ I_{m_1} \otimes (\alpha \otimes D_1) \end{pmatrix}$	$s+1 \le i \le S$, arrival when one server is busy, the other server is idle, and production is off
$(n,i,1)$	$(n+1,i,1)$	$I_{m_1^2 m_2} \otimes D_1$	$n \ge 2, 2 \le i \le S-1$, arrival when both servers are busy and production is on
$(n,i,2)$	$(n+1,i,2)$	$I_{m_1^2 m_3} \otimes D_1$	$n \ge 2, 2 \le i \le S-1$, arrival when both servers are busy and production process is under repair
$(n,i,0)$	$(n+1,i,0)$	$I_{m_1^2} \otimes D_1$	$n \ge 2, s+1 \le i \le S$, arrival when both servers are busy and production is off

Table 3: *Transition table corresponding to no change*

$(0,i,0)$	$(0,i,0)$	D_0	$s+1 \le i \le S$
$(0,i,1)$	$(0,i,1)$	$W \oplus D_0 - V_0$	$0 \le i \le S-1$
$(0,i,2)$	$(0,i,2)$	$U \oplus D_0$	$0 \le i \le S-1$
$(n,0,1)$	$(n,0,1)$	$W \oplus D_0 - V_0$	$n \ge 1$
$(n,0,2)$	$(n,0,2)$	$U \oplus D_0$	$n \ge 1$
$(1,i,1)$	$(1,i,1)$	$I_2 \otimes [T \oplus (W \oplus D_0)] - I_{2m_1} \otimes V_0$	$1 \le i \le S-1$
$(1,i,2)$	$(1,i,2)$	$I_2 \otimes [T \oplus (U \oplus D_0)]$	$1 \le i \le S-1$
$(1,i,0)$	$(1,i,0)$	$I_2 \otimes [T \oplus D_0]$	$s+1 \le i \le S$
$(n,i,0)$	$(n,i,0)$	$T \oplus [T \oplus D_0]$	$n \ge 2, s+1 \le i \le S$
$(n,1,1)$	$(n,1,1)$	$I_2 \otimes [T \oplus (W \oplus D_0)] - I_{2m_1} \otimes V_0$	$n \ge 2, 1 \le i \le S-1$
$(n,1,2)$	$(n,1,2)$	$I_2 \otimes [T \oplus (U \oplus D_0)]$	$n \ge 2, 1 \le i \le S-1$
$(n,i,1)$	$(n,i,1)$	$T \oplus [T \oplus (W \oplus D_0)] - I_{m_1^2} \otimes V_0$	$n \ge 2, 1 \le i \le S-1$
$(n,i,2)$	$(n,i,2)$	$T \oplus [T \oplus (U \oplus D_0)]$	$n \ge 2, 1 \le i \le S-1$

Table 4: *Transition table corresponding to service completions*

From	To	Rate	Description
$(1,i,1)$	$(0,i-1,1)$	$\begin{pmatrix} T^0 \otimes I_{m_2m} \\ T^0 \otimes I_{m_2m} \end{pmatrix}$	$1 \leq i \leq S-1$
$(1,i,2)$	$(0,i-1,2)$	$\begin{pmatrix} T^0 \otimes I_{m_3m} \\ T^0 \otimes I_{m_3m} \end{pmatrix}$	$1 \leq i \leq S-1$
$(1,s+1,0)$	$(0,s,1)$	$\begin{pmatrix} T^0 \otimes \beta_1 \\ T^0 \otimes \beta_1 \end{pmatrix}$	
$(1,i,0)$	$(0,i-1,0)$	$\begin{pmatrix} T^0 \otimes I_m \\ T^0 \otimes I_m \end{pmatrix}$	$s+2 \leq i \leq S$
$(n,1,1)$	$(n-1,0,1)$	$\begin{pmatrix} T^0 \otimes I_{m_2m} \\ T^0 \otimes I_{m_2m} \end{pmatrix}$	$n \geq 2$
$(n,1,2)$	$(n-1,0,2)$	$\begin{pmatrix} T^0 \otimes I_{m_3m} \\ T^0 \otimes I_{m_3m} \end{pmatrix}$	$n \geq 2$
$(2,i,1)$	$(1,i-1,1)$	$\begin{pmatrix} T^0 \otimes I_{m_1m_2m} & I_{m_1} \otimes (T^0 \otimes I_{m_2m}) \end{pmatrix}$	$2 \leq i \leq S-1$
$(2,i,2)$	$(1,i-1,2)$	$\begin{pmatrix} T^0 \otimes I_{m_1m_3m} & I_{m_1} \otimes (T^0 \otimes I_{m_3m}) \end{pmatrix}$	$2 \leq i \leq S-1$
$(2,s+1,0)$	$(1,s,1)$	$\begin{pmatrix} (T^0 \otimes I_{m_1}) \otimes \beta_1 & I_{m_1} \otimes (T^0 \otimes \beta_1) \end{pmatrix}$	
$(2,i,0)$	$(1,i-1,0)$	$\begin{pmatrix} T^0 \otimes I_{m_1m} & I_{m_1} \otimes (T^0 \otimes I_m) \end{pmatrix}$	$s+2 \leq i \leq S$
$(n,2,1)$	$(n-1,1,1)$	$\begin{pmatrix} T^0 \otimes I_{m_1m_2m} & I_{m_1} \otimes (T^0 \otimes I_{m_2m}) \end{pmatrix}$	$n \geq 2$
$(n,2,2)$	$(n-1,1,2)$	$\begin{pmatrix} T^0 \otimes I_{m_1m_3m} & I_{m_1} \otimes (T^0 \otimes I_{m_3m}) \end{pmatrix}$	$n \geq 2$
$(n,i,1)$	$(n-1,i-1,1)$	$(T^0 \otimes \alpha) \otimes I_{m_1m_2m}+$ $I_{m_1} \otimes ((T^0 \otimes \alpha) \otimes I_{m_2m})$	$n \geq 3,$ $3 \leq i \leq S-1$
$(n,i,2)$	$(n-1,i-1,2)$	$(T^0 \otimes \alpha) \otimes I_{m_1m_3m}+$ $I_{m_1} \otimes ((T^0 \otimes \alpha) \otimes I_{m_3m})$	$n \geq 3,$ $3 \leq i \leq S-1$
$(n,s+1,0)$	$(n-1,s,1)$	$((T^0 \otimes \alpha) \otimes I_{m_1}) \otimes \beta_1+$ $I_{m_1} \otimes ((T^0 \otimes \alpha) \otimes \beta_1)$	$n \geq 3$
$(n,i,0)$	$(n-1,i-1,0)$	$(T^0 \otimes \alpha) \otimes I_{m_1m}+$ $I_{m_1} \otimes ((T^0 \otimes \alpha) \otimes I_m)$	$n \geq 3,$ $s+2 \leq i \leq S$

Table 5: *Transition table corresponding to interruptions*

$(0,i,1)$	$(0,i,2)$	$(\gamma \otimes V) \otimes I_m$	$0 \leq i \leq S-1$
$(n,0,1)$	$(n,0,2)$	$(\gamma \otimes V) \otimes I_m$	$n \geq 1$
$(1,i,1)$	$(1,i,2)$	$I_{2m_1} \otimes ((\gamma \otimes V) \otimes I_m)$	$1 \leq i \leq S-1$
$(n,1,1)$	$(n,1,2)$	$I_{2m_1} \otimes ((\gamma \otimes V) \otimes I_m)$	$n \geq 2$
$(n,i,1)$	$(n,i,2)$	$I_{m_1{}^2} \otimes ((\gamma \otimes V) \otimes I_m)$	$n \geq 2, 2 \leq i \leq S-1$

Table 6: *Transition table corresponding to production completions*

$(0,i,1)$	$(0,i+1,1)$	$W^0 \otimes \beta_1$	$0 \leq i \leq S-2$
$(0,S-1,1)$	$(0,S,0)$	$W^0 \otimes I_m$	
$(n,0,1)$	$(n,1,1)$	$\begin{pmatrix} 0 & (\alpha \otimes W^0) \otimes \beta_1 \end{pmatrix}$	$n \geq 1$
$(1,i,1)$	$(1,i+1,1)$	$I_{2m_1} \otimes (W^0 \otimes \beta_1)$	$1 \leq i \leq S-2$
$(1,S-1,1)$	$(1,S,0)$	$I_{2m_1} \otimes (W^0 \otimes I_m)$	
$(n,1,1)$	$(n,2,1)$	$\begin{pmatrix} (\alpha \otimes (I_{m_1} \otimes W^0)) \otimes \beta_1 \\ I_{m_1} \otimes ((\alpha \otimes W^0) \otimes \beta_1) \end{pmatrix}$	$n \geq 2$
$(n,i,1)$	$(n,i+1,1)$	$I_{m_1{}^2} \otimes (W^0 \otimes \beta_1)$	$n \geq 2, 2 \leq i \leq S-2$
$(n,S-1,1)$	$(n,S,0)$	$I_{m_1{}^2} \otimes (W^0 \otimes I_m)$	$n \geq 2$

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Table 7: *Transition table corresponding to repair completion*

$(0,i,2)$	$(0,i,1)$	$U^0 \otimes \beta_1$	$0 \le i \le S-1$
$(n,0,2)$	$(n,0,1)$	$U^0 \otimes \beta_1$	$n \ge 1$
$(1,i,2)$	$(1,i,1)$	$I_{2m_1} \otimes (U^0 \otimes \beta_1)$	$1 \le i \le S-1$
$(n,1,2)$	$(n,1,1)$	$I_{2m_1} \otimes (U^0 \otimes \beta_1)$	$n \ge 2$
$(n,i,2)$	$(n,i,1)$	$I_{m_1 2} \otimes (U^0 \otimes \beta_1)$	$n \ge 2, 2 \le i \le S-1$

Table 8: *Effect of ν on some measures of performance*

ν	$E[N]$	$E[I]$	$E[W]$	b_0	b_1	b_2	p_0	p_1	p_2
1.1	1.7417	4.5946	0.4866	0.3882	0.3500	0.2618	0.2287	0.6461	0.1252
1.2	1.8794	4.4632	0.5888	0.3878	0.3506	0.2615	0.2163	0.6467	0.1370
1.3	2.0449	4.3256	0.7117	0.3875	0.3512	0.2612	0.2039	0.6474	0.1487
1.4	2.2451	4.1814	0.8602	0.3872	0.3519	0.2609	0.1914	0.648	0.1606
1.5	2.4883	4.0305	1.0406	0.3869	0.3524	0.2606	0.1789	0.6486	0.1724
1.6	2.7861	3.8723	1.2613	0.3867	0.3530	0.2603	0.1663	0.6493	0.1844
1.7	3.1537	3.7066	1.5338	0.3864	0.3535	0.2601	0.1537	0.6499	0.1964
1.8	3.6123	3.5330	1.8736	0.3862	0.3540	0.2598	0.1410	0.6506	0.2084
1.9	4.1915	3.3510	2.3028	0.3859	0.3545	0.2596	0.1283	0.6512	0.2205
2.0	4.9349	3.1602	2.8535	0.3857	0.3549	0.2594	0.1155	0.6518	0.2327

Table 9: *Effect of ν on some measures of performance*

ν	p_p	p_u	$E[RI]$	I_0	p_s	p_S
1.1	0.4271	0.2190	0.2409	0.0574	0.4552	0.0463
1.2	0.4283	0.2184	0.2621	0.0655	0.4772	0.0438
1.3	0.4296	0.2178	0.2831	0.0744	0.5000	0.0413
1.4	0.4310	0.2170	0.3038	0.0842	0.5236	0.0388
1.5	0.4324	0.2162	0.3244	0.0949	0.5481	0.0362
1.6	0.4339	0.2154	0.3446	0.1066	0.5734	0.0337
1.7	0.4355	0.2145	0.3646	0.1193	0.5996	0.0311
1.8	0.4371	0.2135	0.3842	0.1333	0.6267	0.0286
1.9	0.4388	0.2124	0.4035	0.1484	0.6547	0.0260
2.0	0.4406	0.2112	0.4224	0.1649	0.6837	0.0234

Table 10: *Effect of k on expected total cost*

k	ETC
1	1479.7
2	1045.0
3	1012.5
4	1040.3
5	1086.1