INFINITE-SERVER QUEUEING SYSTEM WITH WAITING NEGATIVE CUSTOMERS

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Abstract

The paper considers a queueing system with waiting negative customers. The system has two arrival processes: one for positive customers, another for negative ones. In this model, arrived negative customers do not contact with present positive ones but immediately destroy new positive arrivals. To find the joint probability distribution of the number of positive and negative customers, we use the method of asymptotic analysis under the condition of high rate of arrivals. As the result, we derive the approximation of characteristic function of the distribution. Using it, we obtain that one-dimensional stationary probability of the number of positive customers can be approximated by Gaussian distribution. Using numerical evaluations and simulation experiments, we estimate an error and an applicability area of the approximation.

Keywords: queueing system, negative customers, asymptotic analysis

1. Introduction

In today's world, service industries such as customer services, transportation systems, medical facilities and many others are faced with increasing flows of customers, where both positive and negative calls require effective management. From e-commerce to social media, our digital infrastructure is subjected to a constant stream of requests, which can be thought of as customers in a queueing system. In real life, the information technology domain faces various challenges in handling a variety of customers and identifying optimal management strategies. For example, malicious attacks or outages can negatively affect the performance and security of information systems.

Consider the process of processing network requests in a data center. In this system, positive customers are requests for access to specific data or resources that are processed by the data center. Such requests are received continuously in the system for a random time. On the other hand, negative customers can be represented by malicious attacks or DDoS (Distributed Denial of Service) attacks. When such a negative customer arrives, it does not interact with

*The research is supported by Russian Science Foundation according to the research project No.24-21-00454, https://rscf.ru/project/24-21-00454

RT&A, Special Issue, No 8 (85)
Volume 20, July 2025

the previous processed positive customer, but begins its destructive activity of overloading the network infrastructure. When new data access requests arrive, the interference caused by DDoS attacks can block or destroy those requests.

Systems and networks with negative customers were proposed by E. Gelenbe in [1], [2], where he introduced a new class of queueing networks with two types of customers. The first type of customers are regular customers and the server serves them in the usual way. Such customers are called positive or persistent customers. A positive customers obeys the established service and routing rules that determine the dynamics of the network under consideration. On the other hand, the second type of negative customers acts as a signal to induce a positive customers at a node, if any, to leave the node immediately. Such networks were originally used to model biophysical neural networks. In this context, a node represents a neuron. Positive and negative requests routed in the network represent excitation and inhibition signals that increase or decrease in unit potential of the neuron to which they arrive. Extensions of the original Gelenbe network lead to a universal class called G-networks in the literature because it provides a unifying framework for queuing systems and neural networks. This analogy has been discussed in detail in survey articles [3], [4].

The first papers on G-networks and single-node queues with negative customers (G-queues) were written at the end of the 20th century. Thus, the main results were developed in a short period of time. Perhaps because of this novelty and special interest in G-networks, many authors simultaneously worked on similar research topics. The most comprehensive review including more than 300 references was presented in [5].

Queues with negative customers can be used to model failures and packet loss, task completion under speculative parallelism, faulty components in production systems and server breakdowns, and the reaction network of interacting molecules. Negative queries with appropriate destruction discipline extend the modeling capabilities of these queueing models, since real-world phenomena such as failures, packet loss in radio interfaces, load balancing, and disasters can be easily captured. For example, in [6], a retrial queue with positive and negative arrival was considered. Negative arrivals lead to server failures, after which the server should perform a repair. The authors of [7, 8] considered retrial queues in which the negative customers clear the pool of retrial calls. The authors used the method of asymptotic analysis for the study. Queueing systems with catastrophes, server failures, and repairs are considered in [9, 10, 11, 12]. The authors analyzed steady-state behavior of the systems under different conditions.

In [13], an infinite-server queue with negative customers with waiting was considered. Situation when an incoming negative customer instantly destroys a serving positive customer were considered. In contrast to the mentioned work, the presented paper considers the case when an incoming negative customer does not interact with the available positive ones, but waits for the arrival of a new positive customer, destroys it and leaves the system. Let us imagine a system of automatic deployment and rollback of changes in the IT infrastructure. Positive customers are requests to deploy a new version of an application, and negative customers are requests to roll back to the previous version. The deployment server acts as a service appliance. When a deployment request arrives, it occupies the server, the deployment takes place, and the server is released. If a rollback request arrives during the deployment, it 'hangs', waiting for the deployment to complete. Once the new deployment is complete, and a positive customer (to deploy the next change or update) comes in, a negative customer (rollback) "intercepts" this new positive customer, cancels it (preventing the new deployment from starting) and starts the rollback process to the previous version, at which point both customers (rollback and canceled deployment) leave the system.

The rest text is organized as follows. In Sec. 2, mathematical model in the form of a queueing system is proposed, system of Kolmogorov equations for it is constructed. In Sec. 3, we perform asymptotic analysis to find the solution of the system. In Sec. 4, results of numerical experiments are provided. Basing on them, we estimate the accuracy of the obtained approximations and area of their applicability.

2. Mathematical model

2.1. Problem statement

Consider a queueing system with an unlimited number of servers. Let two Poisson flows of customers arrive at the system. The first one with intensity λ delivers normal (positive) customers, the other one with intensity α delivers negative customers. A positive customer arrived at the system when there are no negative customers in it, occupies any free server and immediately starts service within it during exponentially distributed period with parameter μ. We consider a problem with waiting negative customers. This means that when a negative customer arrives, it does not interact with positive customers present in the system but waits for the arrival of a new positive customer. The capacity of the waiting queue for negative customers is unbounded. As soon as a new positive customer arrives, if there are negative customers in the system, one of them destroys this new customer and they both leave the system. Figure 1 shows the scheme of the described system. During the system evolution we may have any combinations of the positive and negative customers presence. For example, some positive customers can arrive when there are no any negative ones in the system. They go to the servers. After that, new negative customers can arrive and they will stay in the system until new positive ones arrive. So, we may have both positive and negative customers being in the system at the same time.

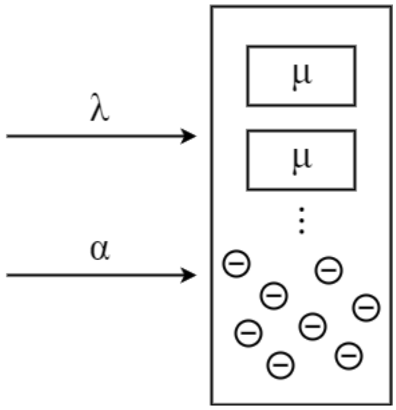

Figure 1: *Queueing system with waiting negative customers*

The paper studies Markov random process $\{i(t), l(t)\}$, where $i(t)$ is the number of positive and $l(t)$ is the number of negative customers in the system at time moment t.

2.2. System of Kolmogorov equations

For probability distribution $P(i,l,t) = \mathrm{P}\{i(t) = i, l(t) = l\}$ of the considered Markov process, we can construct the following system of Kolmogorov differential equations for steady-state regime. For this purpose, let us take time t tending to infinity in probabilities $P(i,l,t)$ and denote $P(i,l) = P(i,l,\infty)$:

$$\begin{gathered}
-(\lambda+\alpha)P(0,0) + \mu P(1,0) + \lambda P(0,1) = 0, \\
-(\lambda+\alpha+i\mu)P(i,0) + \lambda P(i-1,0) + (i+1)\mu P(i+1,0) + \lambda P(i,1) = 0, \\
-(\lambda+\alpha)P(0,l) + \mu P(1,l) + \alpha P(0,l-1) + \lambda P(0,l+1) = 0, \\
-(\lambda+\alpha+i\mu)P(i,l) + \alpha P(i,l-1) + (i+1)\mu P(i+1,l) + \lambda P(i,l+1) = 0.
\end{gathered} \tag{1}$$

Question about the stability condition will be discussed later (see Sec. 3.1).

Let us introduce partial characteristic functions

$$H(u,l) = \sum_{i=0}^{\infty} e^{jui} P(i,l), \; j = \sqrt{-1}, \; l = 0,1,2,\ldots, \; u \in \mathbb{R}.$$

Using the method of partial characteristic functions, system (1) can be written in the form

$$j\mu(1-e^{-ju})\frac{dH(u,0)}{du}+(\lambda e^{ju}-(\lambda+\alpha))H(u,0)+\lambda H(u,1)=0,$$

$$j\mu(1-e^{-ju})\frac{dH(u,l)}{du}-(\lambda+\alpha)H(u,l)+\alpha H(u,l-1)+\lambda H(u,l+1)=0. \quad (2)$$

So, we have obtained system of infinite number of differential equations with variable parameters for infinite number of unknown functions $H(u,l)$, $l=0,1,2,\ldots$ Unfortunately, we do not see ways of its direct solution. Because this reason, we apply a special technique "the asymptotic analysis method" [14] which allows to obtain approximation of the system solution under some conditions.

3. Asymptotic analysis

3.1. First-order asymptotic analysis

Since the direct solution of the obtained system of differential equations is not possible, we apply the method of asymptotic analysis [14] under the condition of the equivalent growing intensities of arrival processes. This condition can be defined as

$$\lambda=\lambda N,\quad \alpha=\alpha N, \quad (3)$$

where we suppose $N\to\infty$. Also, we name this condition as a condition of high intensity of arrivals. The method of asymptotic analysis allows to obtain limit solution of system (2) and use the derived limit expressions for functions $H(u,l)$ as approximations of solution of system (2) when parameter N has big values. So, for numerical evaluations, we take enough big values of N in the obtained expressions (see section 4).

Substituting (3) into (2), we obtain the following system of equations:

$$\frac{1}{N}j\mu(1-e^{-ju})\frac{dH(u,0)}{du}+(\lambda e^{ju}-(\lambda+\alpha))H(u,0)+\lambda H(u,1)=0,$$

$$\frac{1}{N}j\mu(1-e^{-ju})\frac{dH(u,l)}{du}-(\lambda+\alpha)H(u,l)+\alpha H(u,l-1)+\lambda H(u,l+1)=0,\ l\geq 1. \quad (4)$$

Let us denote $\frac{1}{N}=\varepsilon$ and perform substitutions

$$u=\varepsilon w,\qquad H(u,l)=F_1(w,l,\varepsilon).$$

Then system (4) can be rewritten as follows:

$$\varepsilon j\mu(1-e^{-jw\varepsilon})\frac{dF_1(w,0,\varepsilon)}{\varepsilon dw}+(\lambda e^{jw\varepsilon}-(\lambda+\alpha))F_1(w,0,\varepsilon)+\lambda F_1(w,1,\varepsilon)=0, \quad (5)$$

$$\varepsilon j\mu(1-e^{-jw\varepsilon})\frac{dF_1(w,l,\varepsilon)}{\varepsilon dw}-(\lambda+\alpha)F_1(w,l,\varepsilon)+\alpha F_1(w,l-1,\varepsilon)+\lambda F_1(w,l+1,\varepsilon)=0,\ l\geq 1.$$

Consider function $F_1(w,\varepsilon)=\sum_{l=0}^{\infty}F_1(w,l,\varepsilon)$ and sum equations of the system (5). Then we obtain the expression

$$e^{-jw\varepsilon}j\mu\frac{dF_1(w,\varepsilon)}{dw}+\lambda F_1(w,0,\varepsilon)=0.$$

Performing limit transition $\varepsilon\to 0$, we obtain

$$j\mu F_1'(w)+\lambda F_1(w,0)=0, \quad (6)$$

where $F_1(w)=\lim_{\varepsilon\to 0}F_1(w,\varepsilon)$.

Let $\varepsilon \to 0$ in (5). Using notation $F_1(w,l) = \lim_{\varepsilon\to 0} F_1(w,l,\varepsilon)$, we obtain the system of equations

$$\alpha F_1(w,0) + \lambda F_1(w,1) = 0, \tag{7}$$

$$-(\lambda+\alpha)F_1(w,l) + \alpha F_1(w,l-1) + \lambda F_1(w,l+1) = 0,\ l \geq 1.$$

We will seek the solution of this system in the form of $F_1(w,l) = \Phi_1(w)r(l)$, where $\{r(l)\}$ is the stationary probability distribution of the number of negative customers in the system. Substituting this expression into (7), we obtain:

$$\alpha r(0) + \lambda r(1) = 0, \tag{8}$$

$$-(\lambda+\alpha)r(l) + \alpha r(l-1) + \lambda r(l+1) = 0,\ l \geq 1.$$

Solving this system, we obtain

$$r(0) = 1 - \frac{\alpha}{\lambda},$$

$$r(l) = r(0)\left(\frac{\alpha}{\lambda}\right)^l,\ l = 1,2,\ldots$$

We see that system (8) can be solved in terms of probabilities only if $\alpha < \lambda$. This is the stability condition. It also obviously follows from the model: under the opposite inequality, the number of negative customers is increasing unlimitedly.

Using $F_1(w,l) = \Phi_1(w)r(l)$ and taking into account that

$$F_1(w) = \sum_{l=0}^{\infty} F_1(w,l) = \Phi(w)\sum_{l=0}^{\infty} r(l) = \Phi(w),$$

we find $\Phi_1(w)$ from equation (6):

$$j\mu\Phi_1'(w) + \lambda\Phi_1(w)r(0) = 0,$$

$$\frac{d\Phi_1(w)}{dw} = -\frac{\lambda}{j\mu}\Phi_1(w)r(0),$$

$$\frac{d\Phi_1(w)}{\Phi_1(w)} = -\frac{\lambda}{j\mu}r(0)dw,$$

$$\Phi_1(w) = \exp\{j\kappa_1 w\}, \tag{9}$$

where

$$\kappa_1 = \frac{\lambda}{\mu}r(0) = \frac{\lambda-\alpha}{\mu}.$$

3.2. Second-order asymptotic analysis

Taking into account result (9), let us represent functions $H(u,l)$ in the form

$$H(u,l) = H_2(u,l)\exp\{j\kappa_1 uN\}, \tag{10}$$

where $H_2(u,l)$ are some functions ($l = 0,1,2,\ldots$). Substituting (10) into (4), we obtain the following system of equations:

$$\frac{j\mu}{N}(1-e^{-ju})\frac{dH_2(u,0)}{du} - \kappa_1\mu(1-e^{-ju})H_2(u,0)+$$

$$+(\lambda e^{ju} - (\lambda+\alpha))H_2(u,0) + \lambda H_2(u,1) = 0, \tag{11}$$

$$\frac{j\mu}{N}(1-e^{-ju})\frac{dH_2(u,l)}{du} - \kappa_1\mu(1-e^{-ju})H_2(u,l)-$$

$$-(\lambda+\alpha)H_2(u,l) + \alpha H_2(u,l-1) + \lambda H_2(u,l+1) = 0.$$

Let us denote $\frac{1}{N} = \varepsilon^2$ and perform substitutions

$$u = \varepsilon w, \qquad H_2(u,l) = F_2(w,l,\varepsilon).$$

Then we derive

$$\begin{gathered}\varepsilon j\mu(1-e^{-j\varepsilon w})\frac{dF_2(w,0,\varepsilon)}{dw} - \kappa_1\mu(1-e^{-j\varepsilon w})F_2(w,0,\varepsilon)+ \\ +(\lambda e^{j\varepsilon w} - (\lambda+\alpha))F_2(w,0,\varepsilon) + \lambda F_2(w,1,\varepsilon) = 0, \\ \varepsilon j\mu(1-e^{-j\varepsilon w})\frac{dF_2(w,l,\varepsilon)}{dw} - \kappa_1\mu(1-e^{-j\varepsilon w})F_2(w,l,\varepsilon)- \\ -(\lambda+\alpha)F_2(w,l,\varepsilon) + \alpha F_2(w,l-1,\varepsilon) + \lambda F_2(w,l+1,\varepsilon) = 0.\end{gathered} \tag{12}$$

Similarly to [14] and taking into account features of the model under consideration, we will look for solution $F_2(w,l,\varepsilon)$ in the form of expansion

$$F_2(w,l,\varepsilon) = \Phi_2(w)(r(l) + jw\varepsilon g(l)) + O(\varepsilon^2),$$

where $g(l)$ are some unknown function of discrete argument ($l = 0,1,2,...$). Now we substitute this expression into system (12):

$$\begin{gathered}\varepsilon j\mu(1-e^{-j\varepsilon w})\frac{d\Phi_2(w)(r(0)+jw\varepsilon g(0)+O(\varepsilon^2))}{dw} - +\kappa_1\mu(1-e^{-j\varepsilon w})\Phi_2(w)(r(0)+jw\varepsilon g(0)+O(\varepsilon^2))+ \\ +(\lambda e^{j\varepsilon w}-(\lambda+\alpha))\Phi_2(w)(r(0)+jw\varepsilon g(0)+O(\varepsilon^2)) + \lambda\Phi_2(w)(r(1)+jw\varepsilon g(1)+O(\varepsilon^2)) = 0, \\ \varepsilon j\mu(1-e^{-j\varepsilon w})\frac{d\Phi_2(w)(r(l)+jw\varepsilon g(l)+O(\varepsilon^2))}{dw} - +\kappa_1\mu(1-e^{-j\varepsilon w})\Phi_2(w)(r(l)+jw\varepsilon g(l)+O(\varepsilon^2))- \\ -(\lambda+\alpha)\Phi_2(w)(r(l)+jw\varepsilon g(l)+O(\varepsilon^2)) + \alpha\Phi_2(w)(r(l-1)+jw\varepsilon g(l-1)+O(\varepsilon^2))+ \\ +\lambda\Phi_2(w)(r(l+1)+jw\varepsilon g(l+1)+O(\varepsilon^2)) = 0.\end{gathered} \tag{13}$$

In this system, we expand exponents into the Taylor series until power ε^2:

$$\begin{gathered}\varepsilon j\mu(1-(1-j\varepsilon w))(\Phi_2'(w)(r(0)+jw\varepsilon g(0)) + \Phi_2(w)j\varepsilon g(0))- \\ -\kappa_1\mu\left(j\varepsilon w + \frac{(j\varepsilon w)^2}{2}\right)\Phi_2(w)(r(0)+jw\varepsilon g(0))+ \\ +\left(\lambda\left(1+j\varepsilon w+\frac{(j\varepsilon w)^2}{2}\right) - (\lambda+\alpha)\right)\Phi_2(w)(r(0)+jw\varepsilon g(0))+ \\ +\lambda\Phi_2(w)(r(1)+jw\varepsilon g(1)) + O(\varepsilon^3) = 0, \\ \varepsilon j\mu(1-(1-j\varepsilon w))(\Phi_2'(w)(r(l)+jw\varepsilon g(l)) + \Phi_2(w)j\varepsilon g(l))- \\ -\kappa_1\mu\left(j\varepsilon w + \frac{(j\varepsilon w)^2}{2}\right)\Phi_2(w)(r(l)+jw\varepsilon g(l))- \\ -(\lambda+\alpha)\Phi_2(w)(r(l)+jw\varepsilon g(l)) + \alpha\Phi_2(w)(r(l-1)+jw\varepsilon g(l-1))+ \\ +\lambda\Phi_2(w)(r(l+1)+jw\varepsilon g(l+1)) + O(\varepsilon^3) = 0.\end{gathered} \tag{14}$$

Let us consider in (14) terms of different powers of ε. Taking into account only terms without ε, we obtain

$$\begin{gathered}-\alpha r(0) + \lambda r(1) = 0, \\ -(\lambda+\alpha)r(l) + \alpha r(l-1) + \lambda r(l+1) = 0,\end{gathered} \tag{15}$$

which coincides with system (8).

Taking into account only terms with ε in system (14), we obtain the following system of the inhomogeneous finite-difference equations with respect to the discrete-argument function $g(l)$:

$$-\alpha g(0) + \lambda g(1) = r(0)(\lambda - \kappa_1 \mu), \tag{16}$$

$$-(\lambda + \alpha) g(l) + \alpha g(l-1) + \lambda g(l+1) = \kappa_1 \mu r(l).$$

Consider the corresponding homogeneous system of equations

$$-\alpha g(0) + \lambda g(1) = 0,$$

$$-(\lambda + \alpha) g(l) + \alpha g(l-1) + \lambda g(l+1) = 0.$$

The general solution of this system has the form

$$g(l) = C\left(\frac{\alpha}{\lambda}\right)^l,$$

where C is an arbitrary constant, which we find from the boundary condition

$$\sum_{l=0}^{\infty} g(l) = C \sum_{l=0}^{\infty} \left(\frac{\alpha}{\lambda}\right)^l = C \cdot \frac{1}{1 - \frac{\alpha}{\lambda}} = 1,$$

implying

$$C = 1 - \frac{\alpha}{\lambda}.$$

Then the solution of the inhomogeneous system is the sum of the general solution of the homogeneous system and the partial solution of the inhomogeneous system (16). So, we can write the solution of the inhomogeneous system (16) in the form

$$g(l) = \left(1 - \frac{\alpha}{\lambda}\right)\left(\frac{\alpha}{\lambda}\right)^l + \frac{\kappa_1 \mu}{\alpha}\left(1 - \frac{\alpha}{\lambda}\right)\left(\frac{\alpha}{\lambda}\right)^l. \tag{17}$$

Let us consider terms of system (14) which has power ε^2:

$$\Phi_2'(w) r(0) - \kappa_1 w \Phi_2(w) g(0) - \kappa_1 \frac{w}{2} \Phi_2(w) r(0) +$$

$$+ \frac{\lambda}{\mu} \Phi_2(w) \frac{w}{2} r(0) + \frac{\lambda}{\mu} w \Phi_2(w) g(0) = 0,$$

$$\Phi_2'(w) r(l) - \kappa_1 w \Phi_2(w) g(l) - \kappa_1 \frac{w}{2} \Phi_2(w) r(l) = 0, \ l \geq 1.$$

Let us summarize the equations of this system. Denoting

$$G = \sum_{l=0}^{\infty} g(l) = \sum_{l=0}^{\infty} \left(1 - \frac{\alpha}{\lambda}\right)\left(\frac{\alpha}{\lambda}\right)^l + \frac{\kappa_1 \mu}{\alpha}\left(1 - \frac{\alpha}{\lambda}\right)\left(\frac{\alpha}{\lambda}\right)^l =$$

$$= 1 + \frac{\lambda - \alpha}{\alpha} \frac{1}{1 - \frac{\alpha}{\lambda}} = 1 + \frac{\lambda}{\alpha},$$

$$\sum_{l=0}^{\infty} r(l) = \sum_{l=0}^{\infty} \left(1 - \frac{\alpha}{\lambda}\right)\left(\frac{\alpha}{\lambda}\right)^l = \left(1 - \frac{\alpha}{\lambda}\right)\left(\frac{1}{1 - \frac{\alpha}{\lambda}}\right) = 1,$$

we obtain:

$$\Phi_2'(w) \sum_l r(l) - \kappa_1 w \Phi_2(w) \sum_l g(l) - \kappa_1 \frac{w}{2} \Phi_2(w) \sum_l r(l) +$$

$$+ \frac{\lambda}{\mu} \Phi_2(w) \frac{w}{2} r(0) + \frac{\lambda}{\mu} w \Phi_2(w) g(0) = 0,$$

$$\Phi_2'(w) - \kappa_1 w \Phi_2(w) G - \kappa_1 \frac{w}{2} \Phi_2(w) + \frac{\lambda}{\mu} \Phi_2(w) \frac{w}{2} r(0) + \frac{\lambda}{\mu} w \Phi_2(w) g(0) = 0.$$

After grouping the terms, we obtain a first-order differential equation with separating variables:

$$\frac{d\Phi_2(w)}{\Phi_2(w)} = w \left[\kappa_1 \left(\frac{1}{2} + \frac{\lambda}{\alpha} \right) + \frac{\lambda}{\mu} \left(\frac{3}{2} \left(1 - \frac{\alpha}{\lambda} \right) \right) \right],$$

which has solution

$$\Phi_2(w) = \exp \left\{ -\frac{\kappa_2 w^2}{2} \right\},$$

where

$$\kappa_1 = \frac{\lambda - \alpha}{\mu}, \qquad \kappa_2 = \kappa_1 \left(\frac{1}{2} + \frac{\lambda}{\alpha} \right) + \frac{\lambda}{\mu} \left(\frac{3}{2} \left(1 - \frac{\alpha}{\lambda} \right) \right). \tag{18}$$

Turning back to functions $H_2(u, l)$, we obtain:

$$H_2(u, l) = F_2(w, l, \varepsilon) = \Phi_2(w)(r(l) + jw\varepsilon g(l)) + O(\varepsilon^2) =$$

$$= \exp \left\{ -\frac{\kappa_2 w^2}{2} \right\} \left(1 - \frac{\alpha}{\lambda} \right) \left(\frac{\alpha}{\lambda} \right)^l = \exp \left\{ -\frac{\kappa_2 \frac{u^2}{\varepsilon^2}}{2} \right\} \left(1 - \frac{\alpha}{\lambda} \right) \left(\frac{\alpha}{\lambda} \right)^l \approx$$

$$\approx \exp \left\{ -\frac{\kappa_2 u^2 N}{2} \right\} \left(1 - \frac{\alpha}{\lambda} \right) \left(\frac{\alpha}{\lambda} \right)^l.$$

Sign $\approx$ is used because ε tends to 0 (this is equivalent to $N \to \infty$) but we take its pre-limit non-zero value to obtain pre-limit expressions for functions $H_2(u, l)$ and $H(u, l)$.

Given the expression (8), we obtain the partial characteristic function of the joint distribution of the number of positive and negative customers:

$$H(u, l) = \exp \left\{ juN\kappa_1 + \frac{(ju)^2 N \kappa_2}{2} \right\} \left(1 - \frac{\alpha}{\lambda} \right) \left(\frac{\alpha}{\lambda} \right)^l,$$

where κ_1 and κ_2 are determined by expressions (18).

If we sum up this expressions over $l = 0, 1, 2, \ldots$, we can obtain the characteristic function of the one-dimensional random process of the number of positive customers in the system in the steady-state regime:

$$H(u) = \exp \left\{ juN\kappa_1 + \frac{(ju)^2 N \kappa_2}{2} \right\}.$$

Thus, the asymptotic stationary probability distribution of the number of positive customers in the system under consideration is Gaussian with mean $N\kappa_1$ and variance $N\kappa_2$.

If we put $u = 0$, we obtain the probability distribution of the number of negative customers in the system:

$$P(l) = \left(1 - \frac{\alpha}{\lambda} \right) \left(\frac{\alpha}{\lambda} \right)^l, \ l \geq 0.$$

That is, the one-dimensional distribution of the number of negative customers in the system in the stationary regime in the specified asymptotic condition is geometric.

4. Numerical examples and estimation of approximation accuracy

To evaluate the accuracy of the obtained approximation and to establish the limits of its applicability, a series of numerical experiments were conducted, in which the asymptotic distributions were numerically compared with the empirical ones obtained as results of simulation modeling. A program for simulation of the model with waiting negative customers was developed in Python.

The following parameter values were chosen for the experiments: $\lambda = 2$, $\alpha = 1$, $\mu = 1$. In this case, $\kappa_1 = 1$ and the asymptotic average number of positive customers in the system equals to N, which is convenient for understanding the applicability area of the results. Figure 2 shows plots of asymptotic probability distributions of the number of positive customers and probability distributions obtained by the simulation model for different values of parameter N.

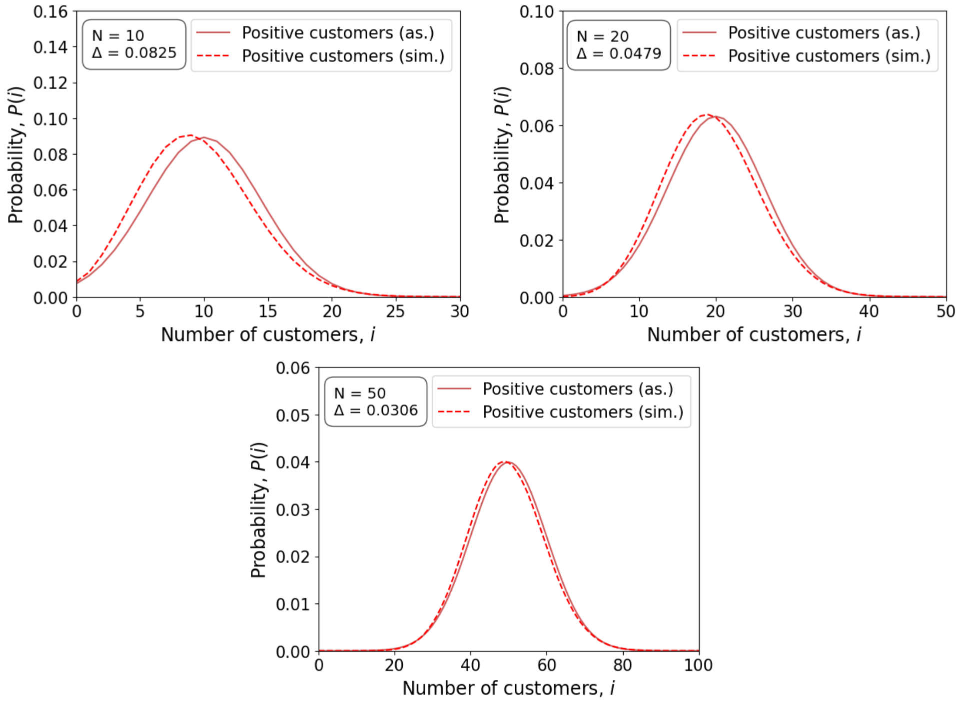

Figure 2: *Comparison of empirical and asymptotic probability distributions of the number of positive customers for various values of parameter N*

To estimate the error of the obtained asymptotic probability distribution of the number of positive and negative customers, let us use the Kolmogorov distance, which is calculated by the formula

$$\Delta = \max_{0 \le i < \infty} \left| \sum_{n=0}^{i} \Big(P_{sim}(n) - P_{as}(n) \Big) \right|,$$

where P_{sim} is the probability distribution obtained from the simulation experiment and P_{as} is the asymptotic probability distribution.

Table 1 presents values of the Kolmogorov distances between the asymptotic distributions of positive customers and the corresponding distributions obtained from the simulations. As we can see from the table and graphs, the Kolmogorov distance decreases while the high-rate parameter N increases. If we choose error $\Delta \le 0.05$ as an acceptable value, then we can conclude that the obtained Gaussian approximation for distribution of positive customers can be applied for $N \ge 20$.

Table 1: *Kolmogorov distance for the distribution of the number of positive customers*

N	10	20	50	100	200	500	1000
Δ	0.0825	0.0479	0.0306	0.0231	0.0172	0.0124	0.0041

Figure 3 shows the plot of the asymptotic probability distribution of the number of negative customers and the corresponding probability distribution obtained from the simulation model for

$N = 100$. The curves almost match, and the Kolmogorov distance is small enough even for small values of asymptotic parameter N (e.g. for $N = 10$, $\Delta = 0.0023$).

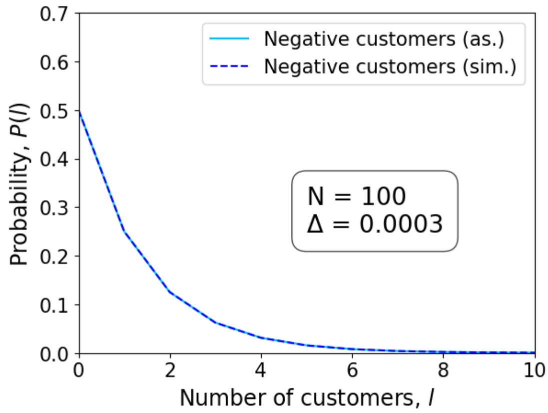

Figure 3: *Comparison of empirical and asymptotic probability distributions of the number of negative customers*

5. Conclusion

In the paper, we consider a mathematical model of a queueing system with waiting negative customers. The asymptotic analysis of this system is performed, the partial characteristic functions of the joint distribution of the number of positive and negative customers are obtained. One-dimensional asymptotic probability distributions of the number of positive and negative customers under the condition of a high rate of arrivals are derived. A series of numerical experiments and comparison with the results of simulation modeling are carried out. Based on these results, we have estimated the accuracy and applicability area of the obtained approximations.

The study may be extended to the models with MAP arrivals or some other non-Poisson processes as well as with an arbitrary distribution of service times.

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