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PERFORMANCE AND NUMERICAL ANALYSIS OF $(GI|GI|N, M)$ QUEUES USING MARKED MARKOV PROCESS

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Abstract

We study the key performance characteristics of a finite-buffer multi-server queuing system denoted as $(GI|GI|n, m)$, with general inter-arrival and service times distributions. The concept called Marked Markov Processes is employed to analyze such a system. Its mathematical model is constructed, and marks' transformations are introduced, which are further applied to calculate the performance characteristics of the system using a special simulation algorithm. Numerical study validates the proposed method employing the comparison of the obtained results with well-known results for $(M|M|1)$, $(M|GI|1)$, and $(M|M|n, m)$ models.

Keywords: $(GI|GI|n, m)$ queuing system, Marked Markov Process, general inter-arrival and service times distributions, steady-state probabilities, stationary performance metrics, numerical analysis.

1. Introduction

Many real stochastic phenomena can be modeled as processes with discrete random intervention (PDRI), first proposed by Kolmogorov. This class of processes is discussed in [1] and some other works. Jump-wise Markov, semi-Markov, Generalized Semi-Markov Processes (GSMP) are notable examples. In the paper, we consider an alternative construction of such processes and demonstrate its application to a widely studied queuing system (QS) with renewal input, buffer capacity m, and n servers with general independent identically distributed (iid) service times. Using Kendall's notation [2], slightly adapted for modern needs in [3], we denote such a system as $(GI|GI|n, m)$.

The initial application of Markov processes in stochastic system analysis focused on processes with discrete (finite or countable) state spaces before being extended in various directions, including, in particular, the method of supplementary variables [4], linear Markov processes [5], and piecewise linear Markov processes [6]. Furthermore, since the publication of Cinlar's paper [7], semi-Markov processes have gained widespread acceptance in the field.

Since Smith's classic paper [8], the regenerative approach has played a pivotal role in studying stochastic systems. Also, it has been established that a wide class of stochastic processes possess a regenerative structure (see, for instance, [9, 10, 11]). This approach proves particularly powerful for analyzing stochastic processes describing complex queuing models.

A critical challenge in studying stochastic systems is that initial information is rarely known with high precision. Thus, analyzing the sensitivity of model output characteristics to the shape and parameters of input distributions is essential. In this context, we highlight:

(i) classic study by Sevastyanov [12] on the insensitivity of the Erlang loss system steady state distribution to the shape of service time distribution, as well as

(ii) the works by Gnedenko [13, 14] and Soloviev [15] on the convergence of the reliability function of the double cold-standby system to the exponential one, regardless of the life/repair time distributions of elements. These findings can be interpreted as the asymptotic insensitivity of the system reliability function to component characteristics. Later on these studies have been extended in a series of our works [16, 17, 18, 19] to a broader class of the reliability systems.

In recent publications, a concept of Discrete Event System (DES) modeling [20, 21, 22, 23] and its mathematical formulation as GSMP [24, 25] has been proposed. These studies demonstrate that GSMPs effectively describe models in queuing theory, reliability theory, and stochastic networks. Moreover, it was shown that GSMPs exhibit a regenerative structure, enabling proofs of the Law of Large Numbers (LLN) and Central Limit Theorem (CLT) in [26, 27]. This regenerative framework offers significant potential for the DES modeling and simulation [20, 21, 22, 23, 24]. While the classical results have been applied mainly to the finite-state processes, the recent research identify the regenerative structures in models with more general state spaces.

Some articles [28, 29, 30] present research results, including numerical, of specific models of stochastic systems based on DES using GSMP. In this paper we propose another mathematical model of PDRI.

In contrast, inspired by the Marked Point Processes, in [31] the concept of the Marked Markov Process (MMP) has been proposed to analyze a hot-standby double system. The subsequent works [32, 33] use this approach to analyze a repairable k-out-of-n systems with general life and repair times distributions of the components. This approach enables both computation of the key reliability metrics and also a sensitivity analysis with respect to the input data.

This paper employs MMPs to study the $(GI|GI|n, m)$ system with focus on the development of the methods and procedures to calculate the key metrics of the system. We mention a few monographs [10, 11, 34, 35, 36] where different queueing systems are considered in various aspects at a fairly advanced mathematical level. In this regard we note that the monograph [22] which also focuses on the algorithms to calculate the performance metrics of the stochastic systems is being the most close to the current paper.

The paper is organized as follows. A general description of the MMP is given in Section 2. In Section 3 the description of the system $(GI|GI|n, m)$, the key notations are given. Moreover, the basic assumptions and the problem setting are presented in this section. A detailed description of the transformations of marks and the calculation of the corresponding transition probabilities are given in Section 4. In Section 5, the analytical expressions for the marks distributions are deduced. The stability conditions are given in Section 6. By the complexity of the computation procedures, Section 7 presents the main formulae expressing the key metrics in terms of the marks. Section 8 contains various numerical examples, which include validation of the proposed methodology. Appendices A and B contain auxiliary information and a general simulation-based algorithm to calculate the performance metrics.

2. Marked Markov Process

Most of all mentioned in the Introduction processes are PDRI. A fairly general mathematical model of PDRI is the Marked Markov Process. By MMP we understand a discrete-time sequence

$$Z := Z(t) = \{(J(t), \mathbf{V}(t)),\ t = 0, 1, \ldots\},$$

at the times $S(t)$ of random intervention which are calculated with the help of marks and allow investigating of all needed characteristics of the system in continuous time. Here

- $J(t)$ is the main component, which describes the system states with a set of states $\mathcal{J}$ with $|\mathcal{J}| \le \infty$,
- $\mathbf{V}(t)$ is a set of random marks $\mathbf{V}(t) = \{\mathbf{V}_j(t) : j \in \mathcal{J}\}$, which makes the process Z Markov, and, for each j, the mark $\mathbf{V}_j$ takes values in a measurable space $(E_j, \mathcal{E}_j)$.

Such a process is determined by:

- The transition probabilities $p_{ij}(\mathbf{V}_i) = \mathbf{P}\{J(t+1) = j | J(t) = i, \mathbf{V}_i(t)\}$ of the component $J(t)$, which depend on the mark $\mathbf{V}_i(t)$ in state i (at step t);
- The marks transformation operators $\Phi_{ij}(\mathbf{V}_i)$ for the transition from state i to state j, based on the content of the mark $\mathbf{V}_i$ in state i, and
- The sequence of the iid random variables (rv's) ξ_t, $t \in \mathbb{N}_0 := [0, 1, \ldots]$ describing the input data and discrete random interventions. These rv's determine the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ on which all other components of the process are also determined. A detailed description of these data depends on a concrete model and, for the considered case, will be given below.

A complete description of such a process includes also
(i) the initial distribution $\alpha = (\alpha_j : \ j \in \mathcal{J})$ of the main component, where $\alpha_j = \mathbf{P}\{J(0) = j\}$,
(ii) the conditional initial distribution $\mu_j(0, \cdot) = \mathbf{P}\{\mathbf{V}(0) \in \cdot \, | J(0) = j\}$ of the marks $\mathbf{V}(t)$ and
(iii) the distribution $F(\cdot)$ of the rv's ξ_t.

This research does not cover a full theory of the MMP but only illustrates some possibilities to calculate the performance of a general queueing model.

3. Model description

3.1. Assumptions and notations

Consider an n-server QS, denoted by $(GI|GI|n,\ m)$, with buffer capacity m, the renewal input process, and the iid service times. Denote by

- $A_i :$ $(i = 1, 2, \ldots)$ iid inter-arrival times of customers with absolutely continuous cumulative distribution function (cdf) $A(t) = \mathbf{P}\{A_i \le t\}$, probability density function (pdf) $a(t) = A'(t)$, finite mean $\mu_A = \mathbf{E}[A_i] < \infty$ and finite coefficient of variation (CoV) defined as the ratio of the standard deviation σ_A to the mean μ_A, $v_A = \sigma_A / \mu_A < \infty$;
- B_i $(i = 1, 2, \ldots)$ the iid service times with absolutely continuous cdf $B(t) = \mathbf{P}\{B_i \le t\}$, pdf $b(t) = B'(t)$, finite mean $\mu_B = \mathbf{E}[B_i] < \infty$ and finite CoV $v_B = \sigma_B / \mu_B < \infty$.

In what follows, the rv's are denoted by capital Latin letters, $X, Y, \ldots$, and their values are denoted by the corresponding lowercase Latin letters, $x, y, \ldots$. Multidimensional rv's and their values are denoted by the bold letters. Furthermore, for an iid sequence $\{X_i\}$, the rv X denotes the generic element.

3.2. The MMP-based System Modeling

The dynamic of the system $(GI|GI|n, m)$ is described by the following MMP

$$Z := \{Z(t) = (J(t), \mathbf{V}(t)),\ t \in \mathbb{N}_0\},$$

where t is the number (step) of intervention (change of the state of system). The (main) component $J(t) \in \mathcal{J} = \{0, 1, 2, \ldots, n+m\}$ denotes the number of customers in the system (at step t) and

$$\mathbf{V} := \{\mathbf{V}(t) = \{\mathbf{V}_j(t), j \in \mathcal{J}\},\ t \in \mathbb{N}_0\},\ \mathbf{V}_j(t) = (X_j(t),\ \mathbf{Y}_j(t)) \tag{1}$$

is the sequence of marks where, in state j in step t, $X_j(t)$ denotes the residual arrival time and $\mathbf{Y}_j(t) = (Y_j^{(1)}(t), \dots, Y_j^{(j\wedge n)}(t))$ are the residual service times (arranged in an ascending order) and $j \wedge n = \min(j, n)$. (By assumption, $Y_0^{(1)}(t) = \infty$ if $J(t) = 0$.) The changes of the process Z in continuous time occur at the instants

$$S(t) = S(t-1) + T_{J(t)}(t), \tag{2}$$

where the (continuous) time interval $T_{J(t)}(t)$ between the switching of states is

$$T_{J(t)}(t) = \min\left[X_{J(t)}(t),\, Y_{J(t)}^{(1)}(t)\right],\ t \in \mathbb{N}_0,\ S(0) = S(-1) := 0. \tag{3}$$

We note that the dimension of the vector $\mathbf{Y}_j(t)$ depends on $J(t) = j$ but is fixed for fixed j.

The 1st customer arrives in the empty system and starts a new service, $J(0) = 1,\ X_1(0) = A_0,\ Y_1(0) = B_0$. The transition probabilities of $J(t)$ do not depend on t (time-homogeneous process) but depend on the associated marks as follows:

$$\begin{aligned} p_j(\mathbf{V}_j) &= \mathbf{P}\{J(t+1) = j+1 \,|\, J(t) = j\} = \mathbf{P}\{X_j(t) \le Y_j^{(1)}(t)\}, \\ q_j(\mathbf{V}_j) &= \mathbf{P}\{J(t+1) = j-1 \,|\, J(t) = j\} = \mathbf{P}\{X_j(t) > Y_j^{(1)}(t)\},\ j \in \mathcal{J}. \end{aligned}$$

These transitions are illustrated in Figure 1 and resemble transitions of a typical birth and death process. Note that relation (2) allows to calculate the required metrics in continuous time. To this

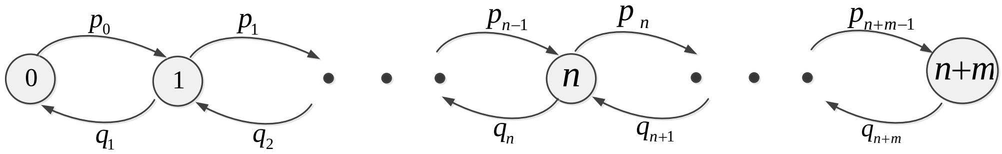


Figure 1: *Transition graph of the process $J(t)$, where $p_j = p_j(\mathbf{V}_j)$ and $q_j = q_j(\mathbf{V}_j)$*

end, we introduce the process

$$N(s) = \min\{t :\ S(t) \le s\},\ s \ge 0,$$

counting (in continuous time) the intervention times in interval $[0, s]$. Then, the number of customers in the system, $L(s)$, in (continuous) time s is defined as

$$L(s) = J(N(s)),\ s \ge 0. \tag{4}$$

If $J(t-1) = 0,\ J(t) = 1$, then t is a regeneration step, and the discrete-time regenerations $\{\tau_i\}$ can be defined by the following recursion,

$$\tau_i = \min\{t :\ t > \tau_{i-1} :\ J(t-1) = 0,\ J(t) = 1\},\ i \ge 1,\ \tau_0 := 0.$$

It is now seen that the instant $S(\tau_i)$ (see (2)) is the beginning of the ith regeneration cycle, that is the ith regeneration point in continuous time. Hence,

$$R_i = S(\tau_i) - S(\tau_{i-1}) = \sum_{\tau_{i-1} < t \le \tau_i} T_{J(t)}(t), \tag{5}$$

is the ith (continuous-time) regeneration cycle length, $i \ge 1$. Figure 2 shows the (right-continuous) trajectory of the process $L(s)$, with the 1st regeneration period $R_1 = S(\tau_1)$. Note that the duration of the ith busy period Π_i is

$$\Pi_i = \sum_{\tau_{i-1} < t \le \tau_i - 1} T_{J(t)}(t) = R_i - T_{J(\tau_i - 1)}(\tau_i - 1),\ i \ge 1. \tag{6}$$

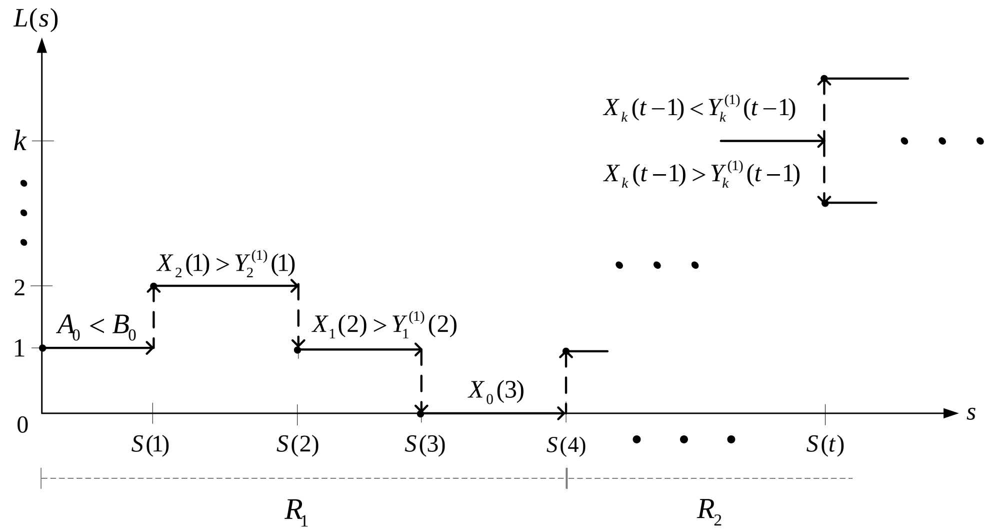


Figure 2: *Trajectory of continuous time process $L(s)$*

Define the number of regenerations in $[0, s]$,

$$K(s) := \min\{k : S(\tau_k) \leq s\}, \ s \geq 0,$$

and assume that the mean regeneration period length $\mu_R := \mathbf{E}[R] < \infty$. Then, by (4), the stationary distribution $\pi_j = \lim_{s\to\infty} \mathbf{P}\{L(s) = j\}$ is calculated, in terms of marks, as follows:

$$\begin{aligned} \pi_j &= \lim_{s\to\infty} \frac{1}{s} \int_0^s 1_{\{L(u)=j\}} du = \lim_{s\to\infty} \frac{1}{s} \left[\sum_{k=1}^{K(s)} \int_{\tau_{k-1}}^{\tau_k} 1_{\{J(N(u))=j\}} du + \int_{\tau_{K(s)}}^{s} 1_{\{J(N(u))=j\}} du \right] = \\ &= \lim_{s\to\infty} \frac{K(s)}{s} \frac{1}{K(s)} \sum_{k=1}^{K(s)} \int_{\tau_{k-1}}^{\tau_k} 1_{\{J(N(u))=j\}} du = \frac{1}{\mu_R} \mathbf{E} \left[\sum_{t=1}^{\tau_1} T_j(t) 1_{\{J(t)=j\}} \right], \ j \in \mathcal{J}. \end{aligned} \tag{7}$$

It is worth mentioning that the regenerative simulation method based on the regenerative version of the CLT allows constructing correct interval estimates, not only the point ones of the performance metrics [23]. However in the numerical analysis in this research we demonstrate the proposed algorithm for the sample-mean estimation while the interval estimation is postponed for a future research.

3.3. The problem setting

The main purpose of the paper is to study the basic stationary performance metrics of the system $(GI|GI|n, m)$ using the MMP framework. Relations (5)-(7) as well as the regenerative structure of the process Z allow to use one (say the 1st) regeneration cycle to calculate the stationary metrics including:

- cdf $F_\Pi(s) = \mathbf{P}\{\Pi \leq s\}$ and the rth moments $m_r(\Pi) = \mathbf{E}[\Pi^r]$, $r > 0$;
- cdf $F_R(s) = \mathbf{P}\{R \leq s\}$ and the rth moments $m_r(R) = \mathbf{E}[R^r]$, $r > 0$;
- stationary distribution π_j, $j \in \mathcal{J}$;
- the loss probability $\pi_{loss} = \pi_{n+m}$ (if $m < \infty$);
- the rth moments of the stationary number of customers $m_r(L) = \sum_{0\leq j\leq n+m} j^r \pi_j$;

- the rth moments of the queue length $m_r(Q) = \sum_{n\le j\le n+m}(j-n)^r\pi_j$;
- the moments of the stationary waiting time W and sojourn time V.

4. Marks Transformations and Distribution

4.1. Transformations of Marks

Consider the sequence of the marks $\{\mathbf{V}_n\}$ and a rv V. Denote by $Sh[\mathbf{V}_n, V]$ the operator subtracting the rv V from the variation series $\mathbf{V}_n$ provided $V \le V^{(1)}$. (If $V = V^{(1)}$, then the vector $\mathbf{V}$ is shifted by one component to the left.) Also denote by $Ad[\mathbf{V}_n,\, V]$ the operator adding an independent rv V to the variation series $\mathbf{V}_n$. More exactly, these operators are defined as

$$\begin{aligned} Sh[\mathbf{V}_n,\, V] &= V^{(i)} - V, \text{ for } i=\overline{1,n}; \qquad (8)\\ Ad[\mathbf{V}_n,\, V] &= \begin{cases} V^{(i)} & \text{for } i<l,\\ V & \text{for } i=l,\\ V^{(i+1)} & \text{for } i>l, \end{cases} \qquad (9)\end{aligned}$$

where $l = \max\{i:\ V^{(i)} \le V\}$. Introduce the operators that transform the process J from state j to $j+1$ and $j-1$, respectively,

$$\mathbf{V}_{j+1} = \Phi_{j,j+1}[\mathbf{V}_j] =: \Phi_j[\mathbf{V}_j], \quad \mathbf{V}_{j-1} = \Phi_{j,j-1}[\mathbf{V}_j] =: \Psi_j[\mathbf{V}_j].$$

Now we prove the following statement.

Statement 1. Under transition $j \to j+1$, i.e. if $X_j(t) \le Y_j^{(1)}(t)$, the marks are transformed as follows:

$$X_{j+1}(t+1) = \Phi_j[X_j(t)] = A_{t+1}. \qquad (10)$$

For $j<n,\ i=\overline{1,n-j}$:

$$\begin{aligned} Y_{j+1}^{(i)}(t+1) &= \Phi_j[\mathbf{Y}_j(t)] = Ad[Sh[\mathbf{Y}_j(t), X_j(t)],\, B_{t+1}] = \\ &= \begin{cases} Y_j^{(i)}(t) - X_j(t) & \text{for } i<l,\\ B_{t+1} & \text{for } i=l,\\ Y_j^{(i+1)}(t) - X_j(t) & \text{for } i>l, \end{cases} \qquad (11)\end{aligned}$$

where $l = \max\{i:\ Y_j^{(i)}(t) - X_j(t) \le B_{t+1}\}$;
for $j \ge n$

$$Y_{j+1}^{(i)}(t+1) = \Phi_j[\mathbf{Y}_j(t)] = Sh[\mathbf{Y}_j(t), X_j(t)] = Y_j^{(i+1)}(t) - X_j(t) \ \ (i=\overline{1,n-j}),\, j\in\mathcal{J}. \qquad (12)$$

Analogously, under transition $j \to j-1$, i.e. if $X_j(t) > Y_j^{(1)}(t)$, then mark X_{j-1} is transformed as

$$X_{j-1}(t+1) = \Psi_j[X_j(t)] = Sh[X_j(t), Y_j^{(1)}(t)] = X_j(t) - Y_j^{(1)}(t),\, j\in\mathcal{J}. \qquad (13)$$

However, the transformation of the mark $\mathbf{Y}_j$ depends on state j and has the following form:
for $j \le n$

$$Y_{j-1}^{(i)}(t+1) = \Psi_j[\mathbf{Y}_j(t)] = Sh[\mathbf{Y}_j(t),\, Y_j^{(1)}(t)] = Y_j^{(i+1)}(t) - Y_j^{(1)}(t) \ \ (i=\overline{1,j}); \qquad (14)$$

and for $j > n$,

$$\begin{aligned} Y^{(i)}_{j-1}(t+1) &= \Psi_j[\mathbf{Y}_j(t)] = Ad[Sh[\mathbf{Y}_j(t),\, Y^{(1)}_j(t)],\, B_{t+1}] \\ &= \begin{cases} Y^{(i)}_j(t) - Y^{(1)}_j(t) & \text{for } \ i < l, \\ B_{t+1} & \text{for } \ i = l, \\ Y^{(i+1)}_j(t) - Y^{(1)}_j(t) & \text{for } \ i > l, \end{cases} \end{aligned} \tag{15}$$

where $l = \max\{i : \ Y^{(i)}_j(t) - Y^{(1)}_j(t) \le B_{t+1}\}; \ j \in \mathcal{J}$.

Comments. Indeed, the transition $j \to j+1$ at instant t occurs when a new customer arrives, then the mark is $X_{j+1}(t+1) = A_{t+1}$, implying (10). To find the change of the mark $\mathbf{Y}_j$, we note that the residual service times $Y^{(i)}_j(t)$ are decreased by $X_j(t)$. Further, if there are available servers ($j < n$), a new customer (with the service time B_{t+1}) takes the ith position in the mark $\mathbf{Y}_{j+1}(t)$. At that, the marks $Y^{(i)}_{j+1}(t)$ preceding the $(i-1)$th position remain unchanged and the marks in the subsequent positions are shifted by one position to the right, it implies (11). If $j \ge n$, then the new customer is queued, if $j < n+m$, and lost, otherwise. At that, the value $X_j(t)$ is subtracted from all components $Y^{(i)}_{j+1}(t)$ in the mark $\mathbf{Y}_j(t)$, and formula (12) follows.

Similarly, the transition $j \to j-1$ is only possible when a service is finished. Then the residual time $X_j(t)$ decreases by $Y^{(1)}_j(t)$, implying (13). Finally, the transformation of $\mathbf{Y}_j(t)$ caused by the transition $j \to j-1$ depends on the state of the system. Namely, for $j \le n$, the components of $\mathbf{Y}_j(t)$ are shifted by $Y^{(1)}_j(t)$, and the value $Y^{(1)}_j(t)$ is removed from the (vector) mark $\mathbf{Y}_j(t)$, proving (14). Otherwise, the 1st awaiting customer joins service, and then $\mathbf{Y}_j(t)$ is transformed according to (15).

It is easy to see that, because the rv's A_t and B_t are independent of the pre-history of the process Z, then the process Z turns out to be Markovian.

Remark 1. When a new customer arrives in the idle system in step t, then $J(t) = 1$ and the new marks take (independent) values $X_1(t) = A$, $Y^{(1)}_1(t) = B$. In other words, step t and the instant $S(t)$ are the regeneration step and the regeneration instant, respectively.

4.2. Transition probabilities of marks transformation

Based on transformations of the marks $\mathbf{V}_j = (X_j, \mathbf{Y}_j)$, we extend the action of operators Sh and Ad to the (non-random) values $\mathbf{v}_j = (x_j,\, \mathbf{y}_j)$. Namely, we put

$$Sh[\mathbf{v}_n,\, v] = v^{(i)} - v, \ \text{ when } \ i = \overline{1, n}; \tag{16}$$

$$Ad[\mathbf{v}_n,\, v] = \begin{cases} v^{(i)} & \text{for } i < l, \\ v & \text{for } i = l, \\ v^{(i+1)} & \text{for } i > l, \end{cases} \tag{17}$$

where $l = \max\{i : \ v^{(i)} < v\}$. Denote transition probabilities by

$$\begin{aligned} P_j(\mathbf{v}_j;\, C_{j+1}) &= \mathbf{P}\{X_j \le Y^{(1)}_j,\ \mathbf{V}_{j+1} \in C_{j+1} | \mathbf{V}_j = \mathbf{v}_j\}; \\ Q_j(\mathbf{v}_j;\, C_{j-1}) &= \mathbf{P}\{X_j > Y^{(1)}_j,\, \mathbf{V}_{j-1} \in C_{j-1} | \mathbf{V}_j = \mathbf{v}_j\}, \end{aligned}$$

where C is a subset of the state space of marks (to be specified below). Under transitions of the main component, the values $\mathbf{v}_j = (x_j, \mathbf{y}_j)$ are transformed by Theorem 1, and we need the following notation

$$\Delta^i_j Sh[\mathbf{y}_j,\, x_j] = (y^{i-1}_j - x_j,\, y^i_j - x_j],\ \Delta^i_j B(Sh[\mathbf{y}_j,\, x_j]) = B(y^i_j - x_j) - B(y^{i-1}_j - x_j),\ i = \overline{2, n-j},\, j \in \mathcal{J}$$

$$\Delta_j^i Sh[\mathbf{y}_j,\, y_j^1] = (y_j^{i-1} - y_j^1,\, y_j^i - y_j^1],\ \Delta_j^i B(Sh[\mathbf{y}_j,\, y_j^1]) = B(y_j^i - y_j^1) - B(y_j^{i-1} - y_j^1),\ i = \overline{2, j},\ j \in \mathcal{J}.$$

Now we define the set $C_{j+1}^{(i)}(\mathbf{v}_j, x)$ corresponding new arrival implying transition $j \to j+1$, (provided $j < n$). This set consists of the values $\mathbf{v}_{j+1}$ (of mark $\mathbf{V}_{j+1}$) for which the component $x_{j+1} \in [0, x]$, while the value $\mathbf{y}_j$ (of mark $\mathbf{Y}_j$), after being shifted by x_j, is complemented by a new component at position i. More exactly,

$$C_{j+1}^{(i)}(\mathbf{v}_j,\, x) = \{\mathbf{v}_{j+1} :\ Ad\big[Sh[\mathbf{v}_j,\, x_j],\ A_{t+1} \in [0, x],\ B_{t+1} \in \Delta_j^i(Sh[\mathbf{y}_j,\, x_j])\big]\}.$$

Similarly, for $j \geq n$, the set $C_{j+1}(\mathbf{v}_j, x)$ contains the values $\mathbf{v}_{j+1}$ with the component $x_{j+1} \in [0, x]$. The new customer is either queued or lost, resulting in shift $\mathbf{Y}_j$ by x_j. Namely,

$$C_{j+1}(\mathbf{v}_j, x) = \{\mathbf{v}_{j+1} :\ A_{t+1} \in [0, x],\ Sh[\mathbf{y}_j,\, x_j]\}.$$

Assume $j \leq n$ and a customer finishes service implying transition $j \to j-1$. Then we define the set $C_{j-1}(\mathbf{v}_j)$ which contains the values of the mark $\mathbf{v}_{j-1}$ shifted by y_j^1, i.e.,

$$C_{j-1}(\mathbf{v}_j) = \{\mathbf{v}_{j-1} :\ Sh[\mathbf{v}_j,\, y_j^1]\}.$$

Similarly, for $j > n$, the set $C_{j-1}^{(i)}(\mathbf{v}_j)$ contains values $\mathbf{v}_j$ shifted by y_j^1. In this case $\mathbf{Y}_j$, is also complemented by the service time B_{t+1} of the customer taken from the ith position in the queue. Formally,

$$C_{j-1}^{(i)}(\mathbf{v}_j) = \{\mathbf{v}_{j-1} :\ Ad\big[Sh[\mathbf{v}_j,\, y_j^1],\ B_{t+1} \in \Delta_j^i(Sh[\mathbf{y}_j,\, y_j^1])\}.$$

The analysis above is summarized in the following statement.

Lemma 1. The kernels of the mark transformations, under transition $j \to j+1$, are: for $j < n$, $i = \overline{2, n-j}$,

$$P_j(\mathbf{v}_j;\, C_{j+1}) \quad = \quad \begin{cases} A(x) \cdot \Delta B_j^i(Sh[\mathbf{y}_j,\, x_j]) & \text{for } \ C_{j+1} = C_{j+1}^{(i)}(\mathbf{v}_j,\, x), \\ 0, & \text{otherwise;} \end{cases}$$

for $j \geq n$, $i = \overline{2, n-j}$,

$$P_j(\mathbf{v}_j;\, C_{j+1}) \quad = \quad \begin{cases} A(x) & \text{for } \ C_{j+1} = C_{j+1}(\mathbf{v}_j,\, x), \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

Analogously, when it goes from state j to state $j-1$ the kernels have the form: for $j \leq n$, $i = \overline{2, j}$,

$$Q_j(\mathbf{v}_j;\, C_{j-1}) \quad = \quad \begin{cases} 1_{\{Ad[\mathbf{v}_j, y_j^1] \in C_{j-1}\}} & \text{for } \ C_{j-1} = C_{j-1}(\mathbf{v}_j), \\ 0, & \text{otherwise;} \end{cases} \tag{19}$$

and for $j > n$, $i = \overline{2, j}$

$$Q_j(\mathbf{v}_j;\, C_{j-1}) \quad = \quad \begin{cases} \Delta B_j^i(Sh[\mathbf{y}_j, y_j^1], e_j(1)]) & \text{for } \ C_{j-1} = C_{j-1}(\mathbf{v}_j), \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

Using these expressions, in the next section, we will consider the distribution of the marks.

5. Distributions of marks

Denote by $\mu_j(t,\cdot) = \mathbf{P}\{\mathbf{V}_j(t) \in \cdot\}$ the measure (distribution) of the mark $\mathbf{V}_j(t) = (X_j(t), \mathbf{Y}_j(t))$ in step $t \in \mathbb{N}_0$, $j \in \mathcal{J}$. Based on the kernels P_j and Q_j, we may write down the equations for the measures $\mu_j(t,\cdot)$

$$\mu_j(t+1,\cdot) = \int \mu_{j-1}(t, \mathbf{dv}_{j-1}) P_{j-1}(\mathbf{v}_{j-1}\,\cdot) + \int \mu_{j+1}(t, \mathbf{dv}_{j+1}) Q_{j+1}(\mathbf{v}_{j+1},\cdot),$$

which, in the terms of operators, can be expressed as follows:

$$\mu_j(t+1,\cdot) = \mu_{j-1}P_{j-1}(t,\cdot) + \mu_{j+1}Q_{j+1}(t,\cdot).$$

The following statement holds.

Corollary 1. By assumption A and B follow absolutely continuous distributions, then the measures of marks are also absolutely continuous and

$$\mu_j(t;C) = \int\cdots\int_C f_j(t;x,y^1,\ldots,y^{j\wedge n})dx \prod_{1\le i\le j\wedge n} dy^i,$$

where $f_j(\cdot)$ is the pdf of marks.

Proof. Using the proof by induction, assume that the initial state $J(0) = 0$ (at time $S(0) = 0$) with the mark $X(0) = A_0$. At step $t = 1$ (at time $S(1) = A_0$), $J(1) = 1$ and $X(1) = A_1$, $Y(1) = B_1$. It is shown in lemma 2 (Appendix A) that in this case subtracting and adding operators keep absolute continuity. Therefore, the measures of the marks are also absolutely continuous. ■

Introduce the following notations,

$$\mathbf{y}_j^i = (y_j^1, \ldots y_j^{i-1}, y_j^{i+1}, \ldots y_j^{j\wedge n}) \text{ and } \mathbf{y}_j^i + u = (y_j^1+u, \ldots y_j^{i-1}+u, y_j^{i+1}+u, \ldots y_j^{j\wedge n}+u).$$

Theorem 1. The pdf f_j satisfies the following recursion:

$$f_1(t+1;x,y) = \left[\int\limits_{u\ge 0} f_0(t;u,\infty)du\right] a(x)b(y) + \int\limits_{u\ge 0} f_2(t;x+u,u,y+u)du; \tag{21}$$

for $2 \le j \le n$

$$\begin{aligned} f_j(t+1;x_j,\mathbf{y}_j) &= \sum_{1\le i\le j} a(x_j)b(y_j^i)\left(B(y_j^{i+1}) - B(y_j^{i-1})\right)\int\limits_{u\ge 0} f_{j-1}(t;u,\mathbf{y}_j^i+u)\,du + \\ &+ \int\limits_{u\ge 0} f_{j+1}(t;x_j+u,u,\mathbf{y}_j+u)du; \end{aligned} \tag{22}$$

for $j > n$

$$\begin{aligned} f_j(t+1;x_j,\mathbf{y}_j) &= a(x_j)\int\limits_{u\ge 0} f_{j-1}(t;u,\mathbf{y}_j^i+u)\,du + \\ &+ \sum_{1\le i\le n} b(y_j^i)\left(B(y_j^{i+1}) - B(y_j^{i-1})\right)\int\limits_{u\ge 0} f_{j+1}(t;x_j+u,u,\mathbf{y}_j+u)du. \end{aligned} \tag{23}$$

Also the following additional condition holds:

$$f_0(t+1;x,\infty) = \int\limits_{u\ge 0} f_1(t;x+u,u)du. \tag{24}$$

Proof. We outline the straightforward proof. The state of the process Z such that $J(t+1) = 1$ with the residual time x and the remaining arrival time y may occur if and only if the following points a) and b) hold:

(a) $J(t) = 0$ (with any remaining time u to new arrival) and, by the agreement, the residual service time is infinite; during this step a new customer arrives and his service time is y with the probability $b(y)dy$, while the remaining arrival time is x with the probability $a(x)dx$; this explains the 1st summand in (21),

(b) $J(t) = 2$ with the (arbitrary) minimal residual service time u and the appropriate residual times; during the step the time u is expired, and it explains the 2nd summand in (21).

For $2 \le J(t+1) = j \le n$, the process Z will be in a 'neighborhood' of the point $\mathbf{v}_j = (x_j, \mathbf{y}_j)$ with the probability $f_j(t+1; x_j, \mathbf{y}_j)\, dx_j \mathbf{dy}_j$, if and only if in step t, either:

(a) $J(t) = j-1$ with the probability $f_{j-1}(t; u, \mathbf{y}_j^i + u)\, du \mathbf{dy}_j^i$ in a 'neighborhood' of the point $(u, \mathbf{y}_j^i + u)$, the residual arrival time u is expired and the components of vector $\mathbf{v}$ are shifted by u; the new arrival time is x_j with the probability $a(x_j)dx_j$, the new customer finds free server and takes the service time y_j^i which is added to the shifted vector $\mathbf{y}_j$ (at the position keeping order) with the probability $(B(y_j^{i+1}) - B(y_j^{i-1}))b(y_j^i)dy_j^i$. It explains the 1st term on the right-hand side of (22); or

(b) $J(t) = j+1$ in a 'neighborhood' of the point $(x_j + u, u, \mathbf{y}_j + u)$, with the probability $f_{j+1}(t; x_j + u, u, \mathbf{y}_j + u)dx_j du \mathbf{dy}_j$, and after completion of service after time u, the process switches to state j in the 'neighborhood' of the point $(x_j, \mathbf{y}_j)$. This explains the 2nd term on the right side of (22).

By a similar way, with the evident changes, can be explained formula (23) when all servers are busy, $n < j \le n+m$, in which case new customer joins the queue or lost.

Finally, to explain (24) we note that $J(t+1) = 0$, $X_0(t+1) = x$ and $Y_0(t+1) = \infty$ if and only if $J(t) = 1$ with an arbitrary $Y_0(t) = u$ and an appropriate residual arrival time, and if the service ends before the new arrival. ■

6. Steady-state analysis

In this section we give the positive recurrence (stability) conditions of the Markov (regenerative) process Z.

6.1. Positive recurrence conditions of the process Z

According to the regeneration theory (see, for example, [10]), the stationary distribution of the discrete-time regenerative process $\{Z(t),\ t \ge 0\}$, as $t \to \infty$, exists if $\mathbf{E}[\tau] < \infty$ and the regeneration cycle length τ is aperiodic rv. Because the system under consideration is finite (the number of customers $\le n+m < \infty$) then the key condition $\mathbf{E}[\tau] < \infty$ is satisfied if the following easily verified condition (connecting the predefined input data) holds true:

$$\mathbf{P}\{A > B\} = \int_0^\infty B(x)A(dx) > 0. \tag{25}$$

It is worth noting, that condition (25) is automatically satisfied, for example, for a Poisson input process, as well as for any renewal process where the inter-arrival interval A has an unbounded support. Note that all components of Z regenerate simultaneously when a new customer arrives in an empty system.

In the context of this study, it is useful to mention the regenerative stability analysis of a repairable finite system containing n unreliable elements with non-exponential lifetimes studied in the recent paper [37].

Remark 2. The regenerative stability analysis remains basically unchanged if the buffer size is unlimited, $m = \infty$. However, in this scenario, we need the well-known negative drift condition [10, 36]

$$\rho := \frac{\mu_B}{n\mu_A} < 1, \tag{26}$$

in addition to the 'regeneration' condition (25). We note that if $n = 1$, then condition (26) implies (25) and that, because the main component process $J(t)$ is itself regenerative, then (under conditions (25), (26) the stationary distribution $\{\pi_j\}$ exists as well.

If the number of servers $n = \infty$, then condition (25) implies stationarity, provided conditions $\mu_A < \infty$, $\mu_B < \infty$ are fulfilled (see [36]). Note that condition (25) is not redundant, since otherwise a basic process describing the dynamics of this system has no classical regenerations.

Remark 3. It is important to note that, within the framework of the approach, we can also consider the unfinished work process in state $J(t)$,

$$W_{J(t)}(t) = \sum_{1 \le i \le J(t) \wedge n} Y_j^{(i)} + \sum_{1 \le i \le [J(t)-n]^+} B_i,$$

which also regenerates at the moment when a new customer arrives in an empty system.

Remark 4. It follows from the Wald's identity that the mean regeneration cycle length in continuous time is finite $\mathbf{E}[R] = \mathbf{E}[\tau]\mathbf{E}[A] < \infty$ provided $\mathbf{E}[\tau] < \infty$. Because the cdf $A(\cdot)$, being absolutely continuous is non-lattice, then the stationary distribution of the process in continuous time also exists. Moreover in this case the cycle length R is spread-out and the convergence to stationary distribution in total variation holds as well [10].

6.2. Steady-state Equations

If the stationary distributions (measures) of marks exist then they satisfy the following equations,

$$\mu_j(\cdot) = \int \mu_{j-1}(\mathbf{dv}_{j-1}) P_{j-1}(\mathbf{v}_{j-1}, \ \cdot) + \int \mu_{j+1}(\mathbf{dv}_{j+1}) Q_{j+1}(\mathbf{v}_{j+1}, \ \cdot) \ \ (j \in \mathcal{J}),$$

with the evident boundary conditions for $j = 0$ and $j = m + n$. These equations can be rewritten in the operator's form as

$$\mu_j(\cdot) = \mu_{j-1} P_{j-1}(\cdot) + \mu_{j+1} Q_{j+1}(\cdot) \ \ (j \in \mathcal{J}). \tag{27}$$

It is worth mentioning that the kernel $P_j + Q_j$ is the identity transformation expressed as $P_j + Q_j = 1_{\{\mathbf{v}_j \in \cdot\}}$. Furthermore, we can utilize operator notation for transition kernels as

$$\mu_j(P_j(\cdot) + Q_j(\cdot)) = \mu_{j-1} P_{j-1}(\cdot) + \mu_{j+1} Q_{j+1}(\cdot),$$

and obtain the recurrence relation

$$\mu_{j-1} P_{j-1}(\cdot) - \mu_j Q_j(\cdot) = \mu_j P_j(\cdot) - \mu_{j+1} Q_{j+1}(\cdot), \tag{28}$$

with the equation

$$\mu_0(C_0) = \mu_1 Q_1(C_0) \tag{29}$$

for the boundary state $j = 0$. Because $y_0^1 = \infty$ in state $j = 0$, then $P_0(\mathbf{v}_0, C_0) = 1_{\{\mathbf{v}_0 \in \cdot\}}$, and (29) can be rewritten as

$$\mu_0 P_0(C_0) = \mu_1 Q_1(C_0) \quad \text{or} \quad \mu_1 Q_1(C_0) - \mu_0 P_0(C_0) = 0.$$

In turn, the latter result allows us to rewrite relation (28) in the form

$$\mu_{j+1} Q_{j+1}(\cdot) = \mu_j P_j(\cdot) \ \ \forall j \in \mathcal{J}. \tag{30}$$

The results above have rather theoretical interest since analytical solutions of (28), (29) seem inaccessible while the numerical solutions are labor-intensive. For this reason, in the next section, we outline how to calculate the main performance metrics directly using the marks.

7. Calculation of the main performance metrics in terms of the marks

In this section, we show that the main performance metrics can be directly obtained in terms of marks of the process Z. For this we use representation of the process $L(s)$ in continuous time (see Figure 2) and its regenerative structure. The regenerative structure allows constructing confidence interval using a single (large enough) trajectory based on the regenerative version of the Central Limit Theorem [23]. Thus, to estimate the main Quality of Service (QoS) metrics we simulate a large enough number K of the paths of the process Z up to the 1st regeneration time $S(\tau_1)$. Thus (supplying the index k to variables from the k-th cycle) based on the formulas (5 – 7), we obtain the sample-mean estimates of

- the cdf of the busy period and its first moment,

$$\hat{F}_{\Pi}(x) = \frac{1}{K}\sum_{1\le k\le K} 1_{\{\Pi^{(k)}\le x\}} \text{ and } \hat{\mu}_{\Pi} = \frac{1}{K}\sum_{1\le k\le K} \Pi^{(k)};$$

- the cdf of the regeneration period and its first moment,

$$\hat{F}_{R}(x) = \frac{1}{K}\sum_{1\le k\le K} 1_{\{R^{(k)}\le x\}} \text{ and } \hat{\mu}_{R} = \frac{1}{K}\sum_{1\le k\le K} R^{(k)};$$

- the steady state probabilities

$$\hat{\pi}_j = \frac{1}{\hat{\mu}_R}\hat{S}_j^{(k)}(\tau) = \frac{\sum_{1\le k\le K} S_j^{(k)}(\tau)}{\sum_{1\le k\le K} R^{(k)}},$$

  where

$$\hat{S}_j^{(k)}(\tau) = \sum_{1\le t\le \hat{S}^{(k)}(\tau)} \hat{T}_j(t) 1_{\{J(t)=j\}}$$

  is the estimation of the time spent by the process in state j during the 1st regeneration period along the kth trajectory and $\hat{S}^{(k)}(\tau)$ is the estimate of the kth regeneration period length;

- all other mean estimates are calculated by replacing in formulas of Section 3.3 the probabilities π_j by their estimates $\hat{\pi}_j$.

Appendix B presents the Algorithm for calculating the stationary performance metrics of the model. The algorithm constructs the process Z trajectories based on the transformations of marks and regenerative simulation method. For numerical analysis, the programming code in Python has been developed.

8. Numerical examples

In this section, we consider a few numerical examples to compare the performance metrics estimated by the Algorithm and the available analytical expressions for systems $(M|GI|1)$ and $(M|M|n, m)$ (or $M/GI/1$ and $M/M/n/m$ in classical notation [2]). Some additional examples of systems $(GI|GI|n, m)$ are presented as well. In all experiments, $K = 10^5$ realizations (paths) of the Algorithm are used. Each path is time time duration of the 1st regeneration cycle.

We use the standard metrics, μ_A, μ_B and CoV's v_A, v_B, for the inter-arrival and service time, respectively. Table 1 demonstrates the characteristics of the used distributions Gnedenko-Weibull, $GW := GW(\alpha, \beta)$, and Gamma, $\Gamma := \Gamma(\alpha, \beta)$, where α is the shape parameter, β is the scale parameter.

Table 1: *Distributions and their characteristics*

Distribution	$GW(\alpha,\beta)$	$\Gamma(\alpha,\beta)$
pdf	$f(s) = \dfrac{\alpha e^{-(s/\beta)^{\alpha}}(s/\beta)^{\alpha-1}}{\beta}, \; s>0$	$f(s) = \dfrac{\beta^{\alpha} e^{-s/\beta} s^{\alpha-1}}{\Gamma(\alpha)}, \; s>0$
mean	$\mu = \beta\Gamma\left(1+\dfrac{1}{\alpha}\right)$	$\mu = \dfrac{\alpha}{\beta}$
variance	$\sigma^2 = \beta^2\Gamma\left(1+\dfrac{2}{\alpha}\right) - \mu^2$	$\sigma^2 = \dfrac{\alpha}{\beta^2}$
CoV	$v = \dfrac{\sigma}{\mu}$	$v = \dfrac{\sqrt{\alpha}}{\alpha}$
distribution parameters	$\alpha, \beta = \dfrac{\mu}{\Gamma(1+1/\alpha)}$	$\alpha = v^{-2}, \quad \beta = \mu v^2$

8.1. Comparison with analytical results and numerical analysis

The analytical expressions for the mean performance metrics of a $(M|GI|1)$ system can be found, for instance, in [10, 34, 35, 38]. It is known that these expressions depend on the 1st and 2nd moments of service time only, while the cdf F_{Π} of the busy period depends on the service time cdf [34],

$$F_{\Pi}(t) = \int_0^t \sum_{k=1}^{\infty} e^{-\mu_A^{-1}x} \frac{(\mu_A^{-1}x)^{k-1}}{k!} b_{(k)}(x)dx, \tag{31}$$

where $b_{(k)}(x)$ is the k-fold convolution of the service time pdf $b(x)$.

To study the infinite-buffer $(M|GI|1)$ system using the Algorithm (Appendix B) we take buffer size $m = 10^4$. First, consider the service time distribution $B(\cdot) \sim \Gamma$. (We denote this system by $(M|\Gamma|1)$.) The mean inter-arrival time of the Poisson input is $\mu_A = 2$. The mean service time is $\mu_B = \mu_A\rho$, where traffic intensity $\rho = 0.5, 0.8, 0.95$, while $v_B = 0.5, 1, 10$.

Fig. 3 shows the cdf F_{Π}[1] of the busy period where we use: solid lines for $\rho = 0.5$, dashed lines

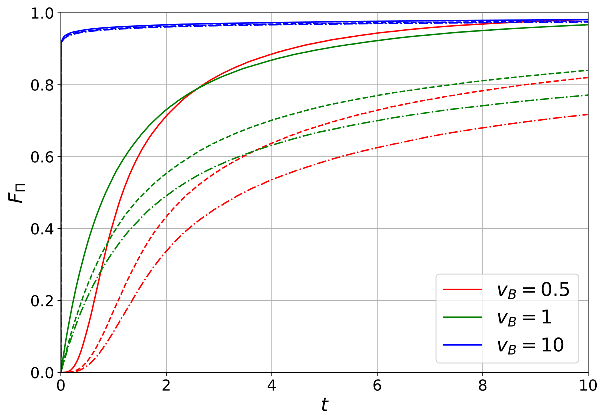


Figure 3: *F_{Π} of the system $(M|\Gamma|1)$ for different values of v_B and ρ*

for $\rho = 0.8$, and dash-dotted lines for $\rho = 0.95$. It is seen that v_B significantly affects F_{Π}. The biggest value of v_B has the minimal impact on F_{Π}, resulting in all curves (black lines) being close to 1. In contrast, for small values ($v_B < 10$), F_{Π} considerably depends on ρ.

Tables 2 and 4 show $\hat{\mu}_{\Pi}$ and $\hat{v}_{\Pi}$ obtained by the algorithm for $B(\cdot) \sim \Gamma$ and GW. Using known analytical formulas (for instance, [10]), $\mu_{\Pi} = \mu_B/(1-\rho)$, then, for $\rho = 0.5, 0.8, 0.95$, it follows

[1]We will use the notation F_{Π} instead of $F_{\Pi}(\cdot)$.

that $\mu_\Pi = 2$, 8, 37.99, respectively. The calculated by the algorithm values of $\hat{\mu}_\Pi$ are close to the analytical ones.

Table 2: *$\hat{\mu}_\Pi$ of the system $(M|GI|1)$ for $B(\cdot) \sim \Gamma\,/\,GW$ (by the algorithm)*

$\hat{\mu}_\Pi$	$\rho = 0.5$	$\rho = 0.8$	$\rho = 0.95$
$v_B = 0.5$	2.0062 / 2.0034	8.0093 / 8.0229	38.2819 / 38.0612
$v_B = 1$	1.9947 / 1.9976	7.9947 / 7.9713	38.5953 / 38.1866
$v_B = 10$	2.0202 / 1.9721	7.9357 / 8.4624	38.8837 / 31.2089

Table 3 presents v_Π calculated, by [35], as

$$v_\Pi = +\sqrt{(\rho + v_B^2)/(1-\rho)}. \tag{32}$$

Table 4 contains values of $\hat{v}_\Pi$ obtained by the algorithm for $B(\cdot) \sim \Gamma/\ GW$ which are consistent

Table 3: *v_Π calculated by (32)*

v_Π	$\rho = 0.5$	$\rho = 0.8$	$\rho = 0.95$
$v_B = 0.5$	1.2247	2.2913	4.8989
$v_B = 1$	1.7321	3.0000	6.2449
$v_B = 10$	14.1774	22.4499	44.9333

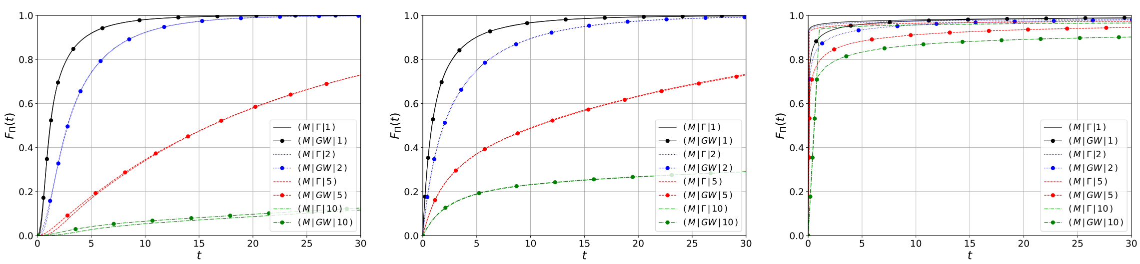
with the results in Table 3. In particular, for fixed ρ, the increase of v_B increases $\hat{v}_\Pi$ as well, while the value of $\hat{v}_\Pi$ is the same, when $B(\cdot) \sim \Gamma$, GW.

Table 4: *$\hat{v}_\Pi$ for $B(\cdot) \sim \Gamma\,/\,GW$ (by the algorithm)*

v_Π	$\rho = 0.5$	$\rho = 0.8$	$\rho = 0.95$
$v_B = 0.5$	1.2271 / 1.2218	2.2682 / 2.2856	4.9411 / 4.9091
$v_B = 1$	1.7394 / 1.7374	2.9561 / 2.9794	6.2628 / 6.5084
$v_B = 10$	13.6669 / 14.0985	22.9887 /21.8648	38.8837 / 35.3533

8.2. Numerical analysis of the system $(M|GI|n)$

In this section, we present the numerical study of n-server system $(M|GI|n)$ with $n = 1, 2, 5, 10$, $\mu_A = 2$, and the fixed traffic intensity $\rho = \dfrac{\mu_B}{n \cdot \mu_A} = 0.5$. It then follows that $\mu_B = 1, 2, 5, 10$. (The case $n = 1$ is included to compare with the earlier obtained results.) Fig. 4 demonstrates F_Π for $B(\cdot) \sim GW$ and Γ.



a) $v_B = 0.5$ b) $v_B = 1$ c) $v_B = 10$

Figure 4: *F_Π for $B(\cdot) \sim \Gamma$ and GW for $n = 1, 2, 5, 10$*

It is seen that F_Π curves are quite close for the given $B(\cdot)$ and v_B for all n, except $v_B = 10$ (fig. 4c). These results demonstrate that the numerical analysis is informative, since illustrates and complements the analytical solutions when they exist. (In this regard compare the results from (31) and the black curves demonstrated F_Π for the $(M|GI|1)$ system.)

As $v_B = 10$, the shape of $B(\cdot)$ plays a significant role and the curves of F_Π for $B(\cdot) \sim GW$ are below than that for $B(\cdot) \sim \Gamma$. For each v_B, F_Π increases as the number of servers n increases. Note that, as $v_B > 1$, the shape parameter $0 < \alpha < 1$ and thus GW service time distribution is heavy-tailed. This observation may be important for a further study of the insensitivity problem.

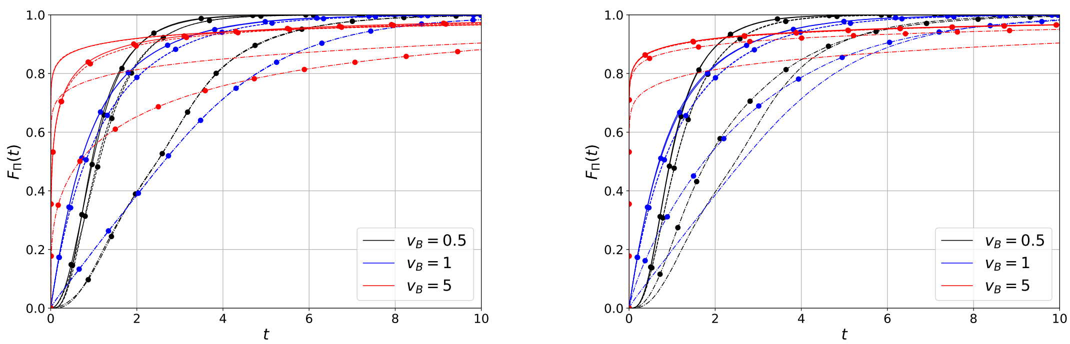
Table 5 shows close values of $\hat{\mu}_\Pi$ for the given cdf's $B(\cdot)$ and v_B if $v_B \leq 1$. For a fixed v_B, increasing n increases $\hat{\mu}_\Pi$ because μ_B increases as well. This interesting observation shows that an increasing μ_B makes performance worse in spite of the increasing the number of servers provided $\rho = 0.5$ is fixed.

Table 5: *$\hat{\mu}_\Pi$ of the system $(M|GI|n)$ for $B(\cdot) \sim \Gamma \,/\, GW$*

$\hat{\mu}_\Pi$	$n = 2$	$n = 5$	$n = 10$
$v_B = 0.5$	4.1047 / 4.1043	23.2388 / 23.1257	297.1719 / 296.7376
$v_B = 1$	4.0125 / 4.0245	22.8843 / 23.0876	295.5491 / 296.3262
$v_B = 10$	4.0356 / 3.8161	23.0863 / 22.1556	305.3809 / 301.4704

8.3. Numerical analysis of the system $(GI|GI|3, 5)$

In this section, we investigate the dependence of the system's metrics $(GI|GI|3, 5)$ on the shape of the inter-arrival time distribution $A(\cdot)$ and its CoV v_A, assuming $\mu_A = 2$, $\mu_B = 2$ and $v_A = v_B = 0.5$, 1, 5. Fig. 5a) demonstrates F_Π of the system $(\Gamma|GI|3, 5)$ for $B(\cdot) \sim \Gamma, GW$ where the curves with markers (with no markers) relate to $B(\cdot) \sim GW$ $(B(\cdot) \sim \Gamma)$. Moreover, we use solid lines for $v_A = 0.5$, dashed lines for $v_A = 1$, and dash-dotted lines for $v_A = 5$.



a) $B(\cdot) \sim \Gamma$ and GW varying v_A, v_B

b) $A(\cdot) \sim \Gamma$ and GW varying v_A, v_B

Figure 5: *F_Π for $(GI|GI|3, 5)$ system*

It is seen that an increase of v_A results in a stochastically decrease of F_Π. At that, for the given cdf $B(\cdot)$, a proximity of the curves with and without markers shows that F_Π weakly depends on the shape of $B(\cdot)$ when $v_B \leq 1$, and strongly depends on both values v_A and v_B (Fig. 5a).

Fig. 5b) shows F_Π for $B(\cdot) \sim \Gamma$ and $A(\cdot) \sim GW, \Gamma$. Here the curves with markers (with no markers) relate to $A(\cdot) \sim GW$ $(A(\cdot) \sim \Gamma)$. It shows a weak dependence of F_Π on the shape of $A(\cdot)$ when $v_A \leq 1$ and $v_B \leq 1$ and if $A(\cdot) \sim \Gamma$ and GW.

Table 6 demonstrates that $\hat{\mu}_\Pi$ strongly depends on both v_A and v_B and changes slightly if $B(\cdot) \sim \Gamma$ and GW. Conversely, v_A and the shape of $B(\cdot)$ have weak impact on $\hat{v}_\Pi$ when $v_A \leq 1$ and $v_B \leq 1$. An increase of v_B implies increasing $\hat{\mu}_\Pi$ and $\hat{v}_\Pi$, while an increase of v_A results in increasing of $\hat{\mu}_\Pi$ and decreasing of $\hat{v}_\Pi$ for fixed v_B.

Table 6: *$\hat{\mu}_\Pi$, $\hat{v}_\Pi$ of the system $(\Gamma|GI|3,5)$ for $B(\cdot) \sim \Gamma/GW$*

	$\hat{\mu}_\Pi$			$\hat{v}_\Pi$		
	$v_B = 0.5$	$v_B = 1$	$v_B = 5$	$v_B = 0.5$	$v_B = 1$	$v_B = 5$
$v_A = 0.5$	1.1340 / 1.1323	1.1407 / 1.1449	1.2421 / 1.1956	0.6444 / 0.6439	1.1528 / 1.1519	5.3147 / 5.3594
$v_A = 1$	1.2955 / 1.3024	1.3032 / 1.3001	1.2925 / 1.3228	0.6537 / 0.6605	1.1176 / 1.1184	5.1876 / 4.9622
$v_A = 5$	2.7021 / 2.6981	3.0914 / 3.0906	3.4152 / 4.6756	0.6236 / 0.6184	0.7811 / 0.7784	3.2407 / 2.8206

Comparing Tables 6 and 7, we may conject that both $\hat{\mu}_\Pi$ and $\hat{v}_\Pi$ slightly depend on the shape of $A(\cdot)$ when $v_A \leq 1$ and $v_B \leq 1$ and for given cdf $A(\cdot)$.

Table 7: *$\hat{\mu}_\Pi$, $\hat{v}_\Pi$ for the system $(GW|\Gamma|3,5)$*

	$\hat{\mu}_\Pi$			$\hat{v}_\Pi$		
	$v_B = 0.5$	$v_B = 1$	$v_B = 5$	$v_B = 0.5$	$v_B = 1$	$v_B = 5$
$v_A = 0.5$	1.1378	1.1535	1.2973	0.6352	1.1369	5.2320
$v_A = 1$	1.2995	1.3050	1.3009	0.6600	1.1193	5.2295
$v_A = 5$	2.3516	2.5265	1.8555	0.7777	1.0115	4.6257

Finally, Table 8 shows that for the given $A(\cdot)$, $B(\cdot)$ the metrics $\hat{\mu}_J$, $\hat{\mu}_Q$, $\hat{\mu}_V$ and $\hat{\mu}_W$ of the system $(GI|GI|3,5)$ change slightly with changing v_B, the shape of $B(\cdot)$, when $v_B \leq 1$, and the shape of $A(\cdot)$, when $v_A \leq 1$. However they strongly depend on v_A, as $v_A \leq 1$.

Table 8: *The performance metrics of the system $(GI|GI|3,5)$*

System performance		$A(\cdot) \sim \Gamma$, $B(\cdot) \sim \Gamma$ / $A(\cdot) \sim \Gamma$, $B(\cdot) \sim GW$			$A(\cdot) \sim GW$, $B(\cdot) \sim \Gamma$		
		$v_B = 0.5$	$v_B = 1$	$v_B = 5$	$v_B = 0.5$	$v_B = 1$	$v_B = 5$
$v_A = 0.5$	$\hat{\mu}_J$	0.5018 / 0.4997	0.4979 / 0.5002	0.5077 / 0.5026	0.5015	0.5001	0.5240
	$\hat{\mu}_Q$	$2 \cdot 10^{-6}$ / $2 \cdot 10^{-6}$	$5 \cdot 10^{-5}$ / $7 \cdot 10^{-5}$	0.0127 / 0.0078	$2 \cdot 10^{-5}$	10^{-3}	0.0143
	$\hat{\mu}_V$	$4 \cdot 10^{-6}$ / $4 \cdot 10^{-6}$	10^{-3} / 10^{-3}	0.0253 / 0.0155	$4 \cdot 10^{-5}$	0.0002	0.0286
	$\hat{\mu}_W$	1.0037 / 0.9993	0.9958 / 1.0003	1.0153 / 1.0053	1.0029	1.0003	1.0481
$v_A = 1$	$\hat{\mu}_J$	0.5019 / 0.5014	0.5047 / 0.5051	0.5148 / 0.5195	0.5000	0.5055	0.5127
	$\hat{\mu}_Q$	0.0021 / 0.0024	0.0031 / 0.0032	0.0177 / 0.0128	0.0021	0.0031	0.0168
	$\hat{\mu}_V$	0.0042 / 0.0047	0.0063 / 0.0063	0.0354 / 0.0257	0.0043	0.0062	0.0337
	$\hat{\mu}_W$	1.0037 / 1.0027	1.093 / 1.0102	1.0296 / 1.0389	1.0000	1.0109	1.0254
$v_A = 5$	$\hat{\mu}_J$	0.5209 / 0.5207	0.5165 / 0.5152	0.5164 / 0.4832	0.6287	0.6166	0.5286
	$\hat{\mu}_Q$	0.1872 / 0.1874	0.1732 / 0.1729	0.0903 / 0.0940	0.1644	0.1502	0.0653
	$\hat{\mu}_V$	0.3743 / 0.3748	0.3465 / 0.3459	0.1805 / 0.1880	0.3288	0.3005	0.1306
	$\hat{\mu}_W$	1.0418 / 1.0415	1.0330 / 1.0304	1.0329 / 0.9664	1.2574	1.2332	1.0573

Note that the case $v_A = v_B = 1$ corresponds to the system $(M|M|3,5)$, and these numerical results are close to that obtained in Table 9 by analytical formulas [38].

Table 9: *The performance metrics of the system $(M|M|3,5)$, analytical results*

m_J	m_Q	m_V	m_W
0.5044	0.0030	0.0060	1.0060

9. Conclusion

The purpose of the current research is to analyze the system $(GI|GI|n, m)$ using the Marked Markov Process and transformations of the marks to compute the main performance indicators of the system. The approach provides the analytical expressions for marks' distributions, as well as conditions for the existence of the stationary distribution of the main process.

Because of the high complexity of the system $(GI|GI|n,m)$, a simulation algorithm is also proposed to evaluate the system's performance metrics. At that, the numerical examples demonstrate consistency with the known analytical expressions. Moreover, the proposed approach allows the simulation-based numerical analysis of the stationary performance indicators, in particular, to study their dependency concerning to given inter-arrival and service time distributions and the coefficients of variation. The obtained in this paper numerical results do not allow making unambiguous conclusions on the sensitivity/insensitivity of the basic performance indicators. However, they show that conclusions based solely on the two first moments of the inter-arrival and service times may lead to inaccurate conclusions of the system performance indicators. Furthermore, the proposed approach applies the simulation directly to construct the trajectories of the process based on the marks transformations, and by our opinion, it opens some new opportunities in stochastic modeling and simulation. In future research, the authors are going to extend the proposed concept of the MMP to consider more complex stochastic systems.

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Appendix A. Auxiliary information

Denote by

$$X^{(1)} \leq X^{(2)} \leq \cdots \leq X^{(i)} \leq \cdots \leq X^{(n)}$$

the variation series for independent sample of rv's X_i $(i = \overline{1,n})$. In order to maintain consistency in the notation for the members of the variation series, we use upper indices in brackets instead of the traditional lower indices. The corresponding rv's values are also denoted with superscripts but without parentheses.

It is well-known (see [39]) that the joint pdf $f(\mathbf{x}) = f(x^1, \ldots, x^n)$ of the variation series $X^{(i)}$ of a sample X_i of independent rv's with pdf $f(x)$ has the form

$$f(x^1, \ldots, x^n) = n! \prod f(x^i) \text{ where } x^1 \leq \cdots \leq x^i \leq \cdots \leq x^n.$$

Lemma 2. Suppose that the pdf $f_{\mathbf{X}_n}(\mathbf{x}_n)$ of the original vector $\mathbf{X}_n$ is continuous in all variables. Then the pdf $f_{Sh[\mathbf{X}_n,X]}(\mathbf{x}_n)$ of the variation series $Sh[\mathbf{X}_n, X]$ obtained by subtracting an independent rv X with the cdf $F_X(x)$ and the pdf $f_X(x)$ satisfying the condition $\mathbf{P}\{X \leq X^{(1)}\} = 1$, is defined by the relation

$$f_{Sh[\mathbf{X}_n,X]}(\mathbf{x}_n) = \int_0^{x^1} f_{\mathbf{X}_n}(\mathbf{x}_n + u) f_X(u) du. \tag{33}$$

In case $X = X^{(1)}$ it holds

$$f_{Sh[\mathbf{X}_n,X]}(\mathbf{x}_{n-1}) = \int_0^{x^1} f_{\mathbf{X}_n}(u, \mathbf{x}_{n-1} + u) du. \tag{34}$$

The pdf $f_{Ad[\mathbf{X}_n,X]}(\mathbf{x}_{n+1})$ of a variation series $Ad[\mathbf{X}_n, X]$ obtained by adding an independent rv X, with the cdf $F_X(x)$ and the pdf $f_X(x)$, has the form

$$f_{Ad[\mathbf{X}_n,X]}(\mathbf{x}_{n+1}) = \sum_{1 \leq i \leq n+1} (F_X(x^i) - F_X(x^{i-1})) f_X(x^i) f_{\mathbf{X}_n}(\mathbf{x}_{n+1} \setminus \{x^i\}), \tag{35}$$

where $x^0 = 0$, $x^{n+2} = \infty$.

Proof. Indeed, the subtracting can be viewed as a non-degenerate linear transformation with a determinant equal to one. Formulas (33, 34) follow by the total probability law taking into account the independence of the rv X. Denote by $d\mathbf{x}_n = dx^1 dx^2 \dots dx^n$. Then, if $X \le X^{(1)}$, we get formula (33):

$$\begin{aligned} f_{Sh[\mathbf{X}_n,X]}(\mathbf{x}_n)d\mathbf{x}_n &= \mathbf{P}\{\mathbf{X}_n - X \in [\mathbf{x}_n, \mathbf{x}_n + d\mathbf{x}_n]\} \\ &= \int_0^{x^1} \mathbf{P}\{\mathbf{X}_n - X \in [\mathbf{x}_n, \mathbf{x}_n + d\mathbf{x}_n] | X \in [u, u+du]\}\mathbf{P}\{X \in [u, u+du]\} \\ &= \int_0^{x^1} \mathbf{P}\{\mathbf{X}_n \in [\mathbf{x}_n + u, \mathbf{x}_n + u + d\mathbf{x}_n]\}\mathbf{P}\{X \in [u, u+du]\} \\ &= \int_0^{x^1} f_{\mathbf{X}_n}(\mathbf{x}_n + u) f_X(u) du\, d\mathbf{x}_n. \end{aligned}$$

If $X = X^{(1)}$, then the variation series $Sh[\mathbf{X}_n, X]$ transforms to vector $(X^{(2)} - X^{(1)}, \dots, X^{(n)} - X^{(1)})$ of dimension $n-1$. The joint pdf of this vector satisfies the following relation,

$$\begin{aligned} & f_{Sh[\mathbf{X}_n,X]}(\mathbf{x}_{n-1})d\mathbf{x}_{n-1} = \mathbf{P}\{X^{(2)} - X^{(1)} \in [x^1, x^1 + dx^1], \dots, X^{(n)} - X^{(1)} \in [x^{n-1}, x^{n-1} + dx^{n-1}]\} \\ & = \int_0^{x^1} \mathbf{P}\{X^{(1)} \in [u, u+du], X^{(2)} - X^{(1)} \in [x^1, x^1 + dx^1], \dots, X^{(n)} - X^{(1)} \in [x^{n-1}, x^{n-1} + dx^{n-1}]\} \\ & = \int_0^{x^1} f_{\mathbf{X}_n}(u, x^1 + u, \dots, x^{n-1} + u) du\, d\mathbf{x}_{n-1} = \int_0^{x^1} f_{\mathbf{X}_n}(u, \mathbf{x}_{n-1} + u) du\, d\mathbf{x}_{n-1}, \end{aligned}$$

which implies relation (34).

Further, when a rv X by is added to the series $\mathbf{X}_n$, this rv takes any position $X^{(i)}$ between the values x^{i-1} and x^{i+1} with probability $F_X(x^{i+1}) - F_X(x^{i-1})$. At that it can also takes the value in a small neighborhood of a point x^i with probability $f_X(x^i)dx^i$. Therefore, representing its joint pdf in terms of probabilities, we have

$$\begin{aligned} & f_{Ad[\mathbf{X}_n,X]}(\mathbf{x}_{n+1})d\mathbf{x}_{n+1} = \mathbf{P}\{X^{(1)} \in [x^1, x^1 + dx^1], \dots, X^{(n+1)} \in [x^{n+1}, x^{n+1} + dx^{n+1}]\} = \\ & = \sum_{1 \le i \le n+1} \mathbf{P}\{x^{i-1} < X \le x^{i+1}\}\mathbf{P}\{X \in [x^i, x^i + dx^i]\}\mathbf{P}\{X^{(j)} \in [x^j, x^j + dx^j],\ j \ne i\} = \\ & = \sum_{1 \le i \le n+1} (F_X(x^{i+1}) - F_X(x^{i-1})) f_X(x^i) f_{\mathbf{X}_n}(\mathbf{x}_{n+1} \setminus \{x^i\}) d\mathbf{x}_{n+1}, \end{aligned}$$

that proves the relation (35) and completes the proof of this lemma. ■

Appendix B. Algorithm

We use simulation to calculate the empirical estimates of the required steady-state indicators of the model. To describe the algorithm we use operators introduced in (16) and apply them to the values of the appropriate arrays,

$$\begin{aligned} Sh[\mathbf{v}_j, v] &\equiv \{v^i_{sh} = v^i_j - v,\ \ (i = \overline{1,j},\ \ v \le v^i_j)\} \\ Ad[\mathbf{v}_j, v] &\equiv \left\{ \begin{array}{l} v^i_{ad}(v) = v^i_j, \text{ as } i < l, \\ v^l_{ad}(v) = v, \\ v^{i+1}_{ad}(v) = v^i_j \text{ as } \ i > l\ \ (i = \overline{1,j-1}), \end{array} \right. \end{aligned}$$

where $l = \max\{i :\ v^i_j < v\}$. Remind that the dimension of the array $\mathbf{y}_j$ is $n+m$.

Remark 5. Remind that due to the regenerative structure of the model it is enough to investigate the process behavior only along one (the first) regenerative period. Thus, the algorithm simulates K trajectories up to the first regeneration point. Note, that in the Algorithm letters t, τ have another sense than in the main text, but their new sense is additionally explained in the Algorithm to avoid misunderstandings. Especially, the letter t_j is used as an estimator for the time spent by the process in state j along the trajectory of the process up to the first regeneration, $t_j = \hat{S}_j^{(k)}(R)$.

Algorithm.

Preparation: Initialize the following initial data: integers n, m; K is the number of model trajectories. Set the distributions $A(\cdot)$, $B(\cdot)$ of rv's A_i, B_i, along with the corresponding mean (μ_A, μ_B) and CoV (v_A, v_B).

Prepare the counters: $(\nu_0, \ldots, \nu_{n+m})$ is the number of visits to system states; $(t_0, \ldots t_{n+m})$ are dwell times in system states; k is the current number of trajectory; τ is the number of loss customers; and an array of length K Π is the time until the system returns to state 0 from state 1 for each kth trajectory.

Beginning. Put $j = 0$, $k = 1$, $\Pi^{(k)} = 0$. Calculate marks in the initial system state $x_0 = A_0$, $\mathbf{y}_0 = \{y_0^1 = \infty\}$.

Step 1. If $k < K$, go to Step 2, if no, go to Step 6.

Step 2. If $j = 0$, calculate $t_j := t_j + x_j$, $\nu_j := \nu_j + 1$, $j := j + 1$. Find $x_j = A_j$, $\mathbf{y}_j = Ad[B]$.
Go to Step 5.

Step 3. While $0 < j < n + m$, repeat:
it $x_j \le y_j^1$:
put $t_j := t_j + x_j$, $\nu_j := \nu_j + 1$, $\Pi^{(k)} := \Pi^{(k)} + x_j$, $j := j + 1$;
calculate $x_j = A_j$;
if $j \le n$, then $\mathbf{y}_j = Ad[Sh[\mathbf{y}_{j-1}, x_{j-1}], B]$
in another case $j > n$, then $\mathbf{y}_j = Sh[\mathbf{y}_{j-1}, x_{j-1}]$
if $x_j > y_j^1$:
put $t_j := t_j + y_j^1$, $\nu_j := \nu_j + 1$, $\Pi^{(k)} := \Pi^{(k)} + y_j^1$, $j := j - 1$;
calculate $x_j = Sh[x_{j-1}, y_{j-1}^1]$;
if $j \le n$, then $\mathbf{y}_j = Sh[\mathbf{y}_{j-1}, x_{j-1}]$
in another case $j > n$, then $\mathbf{y}_j = Ad[Sh[\mathbf{y}_{j-1}, x_{j-1}], B]$

If $j = 0$, then put $y_0^{(1)} = \infty$ and go to Step 2. If $j = n + m$, then go to Step 4. Otherwise, repeat Step 3, while the condition $0 < j < n + m$ is true.

Step 4. If $j = n + m$,
If $x_j \le y_j^1$:
put $\tau := \tau + 1$
calculate $\mathbf{y}_j = Sh[\mathbf{y}_{j-1}, x_{j-1}]$; $x_j = A_j$;
Repeat Step 4 from the beginning.
if $x_j > y_j^1$:
put $t_j := t_j + y_j^1$, $\nu_j := \nu_j + 1$, $\Pi^{(k)} := \Pi^{(k)} + y_j^1$, $j := j - 1$;
calculate $x_j = Sh[x_{j-1}, y_{j-1}^1]$, $\mathbf{y}_j = Ad[Sh[\mathbf{y}_{j-1}, y_{j-1}^1], B]$.
Go to Step 3.

Step 5. Collect statistics:

- Filling counters $\nu_0, \ldots, \nu_{n+m}$,
- Filling the array Π by values $\Pi^{(k)}$,
- Filling counters $t_0, \ldots t_{n+m}$.

Put $k := k + 1$ and go to Step 3.

Step 6. Processing statistics:

- Calculating the distribution of the number ν_j of visits to the states, $\hat{\nu}_j = \dfrac{\nu_j}{\sum_{j \le n+m} \nu_j}$,

- Calculating steady-state probabilities $\hat{\pi}_j = \dfrac{t_j}{\sum\limits_{0\le j\le n+m} t_j}$,
- Calculating cdf of the busy period $\hat{F}_\Pi(x) = \frac{1}{K}\sum\limits_{1\le k\le K} 1_{\{\Pi^{(k)}\le x\}}$;
- Calculating the mean busy period $\hat{\mu}_\Pi = \frac{1}{K}\sum\limits_{1\le k\le K} \Pi^{(k)}$;
- Calculating the loss probability $\hat{\pi}_{loss} = \hat{\pi}_{n+m}$ and loss rate $\hat{\lambda}_{loss} = \mu_A^{-1}\hat{\pi}_{n+m}$,
- Calculating mean queue length $\hat{\mu}_Q = \sum_{n<j\le n+m}(j-n)\hat{\pi}_j$,
- Calculating the mean number of customers in the system $\hat{\mu}_J = \sum_{1\le j\le n+m} j\hat{\pi}_j$,
- Calculating the mean waiting time $\hat{\mu}_W = \mu_A\hat{\mu}_J$ and the mean sojourn time $\hat{\mu}_V = \mu_A\hat{\mu}_Q$.

Stop.