ON THE RELIABILITY ESTIMATION OF THE GAUSSIAN DEGRADATION SYSTEM WITH A PATH-DEPENDENT MEAN DEGRADATION RATE

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Abstract

We consider a system whose degradation dynamic is described by an underlying stochastic process that consists of two components: a centered Gaussian process and a drift term with a so-called path-dependent intensity rate, which means its dependence on the degradation history. The main goal is to estimate the reliability of the system via simulation methods, as its analytical expression is generally not available. The cross-entropy method has been applied to estimate the required quantity with acceptable accuracy. A few numerical experiments have been conducted to study the properties of the proposed estimator.

Keywords: Reliability, Degradation process, Gaussian process, Variance reduction methods, Importance sampling, Cross-entropy method

1. Introduction

The development and evaluation of the models describing the degradation process is an actual research area in reliability analysis since it allows simulation of the damage accumulation process giving an opportunity to estimate the failure probability of the system when its degradation level reaches some critical threshold value.

The degradation arises randomly for many technical systems due to various sources of uncertainties. Thus, it seems quite natural to model the degradation evolution of the system as a stochastic process. There are a few typical classes of stochastic processes that are often used to model the degradation evolution. If the degradation process is monotonic, the Gamma processes [1, 2, 3, 4] and Inverse Gaussian processes [5, 6, 7] (i.e. stochastic processes with independent and stationary increments having Gamma and inverse Gaussian distribution respectively) are often used to model the degradation evolution. Nevertheless, the degradation dynamic of many systems exhibits non-monotonic behavior, that is why the models based on the Wiener process (a well-recognized example of the Gaussian process with stationary and independent increments) are widely used instead [8, 9, 10, 11, 12, 13, 14, 15].

The correlation structure of the degradation process can be rather complicated, thus independence of the increments is not a realistic assumption. Hence, general Gaussian processes whose distributions are completely defined by the mean and covariance function seems a good choice for the degradation modeling [16, 17, 18]. Degradation models based on the Gaussian processes have been successfully applied for different practical issues, such as modeling of the

The standard models assume a fixed mean degradation rate, which is not realistic in practice. The multi-phase Wiener degradation system with a deterministic sequence of change points has been proposed in [15]. The more general case of general Gaussian process with stationary and possibly dependent increments was considered in [21], where the required performance measures were estimated via the Monte Carlo simulation technique using the special variant of the conditional Monte Carlo method to reduce the variance of the estimator. In the next work [22] the case of random change points has been studied and a few variance reduction methods including importance sampling and control variates have been applied.

In this paper, we consider a more general setting when the degradation intensity at each time instant is a random variable whose distribution could depend on the path of the underlying stochastic process, i.e. on the degradation history up to the current time instant. Such an assumption can model possible acceleration or deceleration of the degradation process. In order to estimate the reliability of the considered degradation system we apply the cross-entropy method aiming at the approximation of the so-called zero-variance proposal distribution followed by the standard importance sampling step.

The rest of this paper is organized as follows. Section 2 describes the proposed Gaussian degradation model with a path-dependent degradation intensity. Section 3 is devoted to estimating performance measures of the considered reliability model via Monte Carlo simulation focusing on the variance reduction techniques. The general idea of the cross-entropy method is discussed as well as a few implementation details related to the considered rare-event simulation problem. The results of the numerical experiments are presented in Section 4. Finally, a few concluding remarks are given in Section 5.

2. Model description

The degradation dynamic of the considered reliability system is governed by the stochastic process $\{A(t),\, t \in \mathcal{T}\}$ defined as

$$A(t) = \Lambda(t) + X(t), \tag{1}$$

where the terms on the right-hand side are defined as follows:

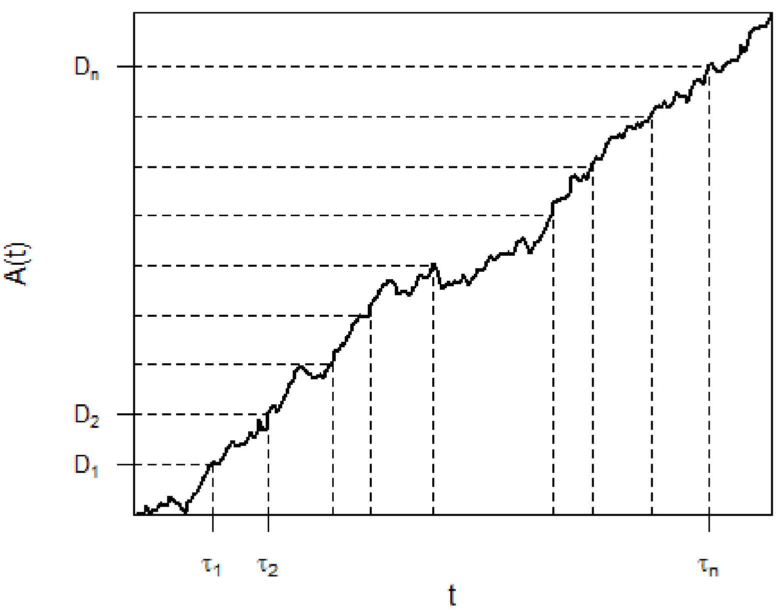

Figure 1: *Path-dependent change-points of degradation intensity.*

- $\{X(t),\ t \in \mathcal{T}\}$ is the centered Gaussian process with a covariance function

$$\Gamma(t,s) := \mathbb{E}\left[X(t)X(s)\right].$$

  The process X can be seen as an additive correlated noise reflecting real-world uncertainties and variations in degradation evolution due to external latent factors.

- The drift term $\Lambda(t) = m(t)t$ has a time-dependent degradation rate $m(t)$, which generally could be a random variable. In this research, it is additionally assumed that $m(t)$ depends on the path of the degradation process (i.e. degradation history) $(A(s),\ s < t)$ up to the current time instant t. Thus, we call it a path-dependent degradation intensity rate.

Now we give the following concrete examples of the path-dependent degradation rates:

1. The path-dependent change-points of the degradation intensity (see. fig. 1):

$$m(t) = \sum_{i \geq 1} m_i \cdot I(\tau_{i-1} < t \leq \tau_i),$$

   where I denotes the indicator function, $0 = \tau_0 < \tau_1 < \tau_2 < \dots$ is a sequence of random variables defined as the successive hitting times

$$\tau_i = \min\{t : \ A(t) \geq D_i\}$$

   with $D_1 < D_2 <$ ... being increasing sequence of the given intermediate thresholds; $\{m_i\}$ are given values. Note that if, at some time instant t, $m(t) = m_i$, then, after instant t, degradation rate can not take any value m_k with $k < i$.

2. The current degradation rate depends on the previous degradation level (see. fig. 2)

$$m(t) = \begin{cases} \sum_{i \geq 1} m_i \cdot I(D_{i-1} < A(t-1) \leq D_i), & t \geq 1, \\ m_0, \quad t < 1, \end{cases}$$

   where $\{m_i\}$ are given values, $0 = D_0 < D_1 < D_2 <$ ... is increasing sequence of the given intermediate thresholds. This example is quite similar to the previous one but allows $m(t)$ return to the previous levels (such an event potentially occur when the degradation process can locally decrease).

Note that the two examples given above illustrate the general idea of the path-dependent intensity rate. We believe that such an assumption reflects different scenarios of the acceleration or deceleration of degradation dynamic depending on the prehistory of degradation evolution. At the same time, more complex models can be considered.

The lifetime of the considered system is defined as follows

$$T_D := \min\{t \in \mathcal{T} : \ A(t) \geq D\}, \tag{2}$$

where D is the given last threshold. We are interested in the estimating the reliability of the system defined as the tail distribution of the lifetime:

$$R(u) := \mathbb{P}(T_D \geq u), \quad u > 0. \tag{3}$$

The closed-form expression of the target performance measure (3) is not available in general except a few simple particular cases of the Wiener degradation model with a deterministic sequence of the degradation intensity change-points (see [15] for more details). Thus, evaluate the tail distribution (3), one has to rely on the Monte Carlo methods which are discussed in the next section.

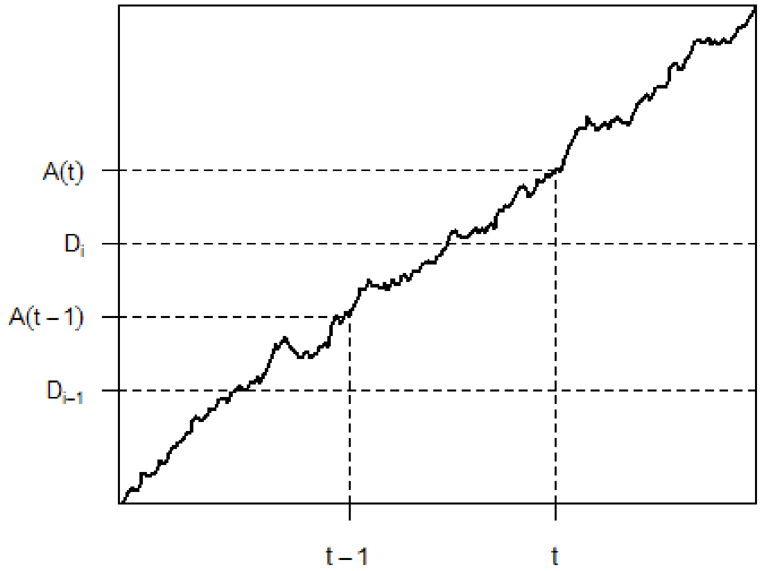

Figure 2: *Dependence of the current degradation rate on the previous degradation level: when $D_{i-1} < A(t-1) \leq D_i$ the current degradation rate is m_i.*

3. Monte Carlo estimation

Denote by Z_u an unbiased estimator of $R(u)$, that is $\mathbb{E}Z_u = R(u)$. Obviously, $R(u) \to 0$ as $u \to \infty$, thus u is called the rarity parameter. To estimate $R(u)$ by the Monte Carlo (MC) simulation, one has to sample from the distribution of the random variable Z_u and calculate the sample mean

$$\widehat{R}_u := \frac{1}{N} \sum_{n=1}^{N} Z_u^{(n)}. \tag{4}$$

The measure of the quality of the estimator is expressed by the *relative error* (RE):

$$\text{RE}\left[\widehat{R}_u\right] := \frac{\sqrt{\mathbb{V}\text{ar}\left[\widehat{R}_u\right]}}{\mathbb{E}\left[\widehat{R}_u\right]}. \tag{5}$$

The standard MC approach is based on the indicator of the target event, i.e.

$$Z_u^{\text{MC}} = I(T_D \geq u).$$

It is straightforward to show that

$$\text{RE}\left[\widehat{R}_u^{\text{MC}}\right] \sim \frac{1}{\sqrt{N \cdot R(u)}}, \quad \text{as} \quad u \to \infty,$$

where $a \sim b$ means $a/b \to 1$. Thus, the RE of the standard MC estimator tends to infinity when the target probability tends to zero, hence a large sample size is required to get a suitable RE. Moreover, in order to have bounded RE the sample size N must grow at least at the same rate as $1/R(u)$ when $u \to \infty$.

There are a few rare event simulation techniques [23, 24] aiming at modifying the estimator (4) to reduce its variance, hence requiring less sample size for the desired accuracy. One class of these methods, namely importance sampling, is briefly discussed below.

In what follows, we restrict ourselves to the finite-dimensional case (enough for the simulation needs) when $\mathcal{T} = \{t_1, ..., t_L\}$, where L is the required simulation length (then t_L is the simulation horizon).

3.1. Importance Sampling

Importance sampling is a widely used method for variance reduction. Its main idea is selecting the proposal distribution so that the target rare event becomes more likely to occur.

Let $f(\mathbf{x})$ be the probability density function (pdf) of the Gaussian random vector $(X(t_1),...,X(t_L))$ and

$$h_u\mathbf{x}) = I(T_D(\mathbf{x}) \geq u), \quad \mathbf{x} \in \mathbb{R}^L.$$

Note that

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{L/2}|\mathbf{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}\mathbf{x}^T\mathbf{\Sigma}^{-1}\mathbf{x}\right), \quad \mathbf{x} \in \mathbb{R}^L, \tag{6}$$

where the covariance matrix

$$\mathbf{\Sigma} = ||\Gamma(t_i,t_j)||_{i,j=1,...,L}. \tag{7}$$

Having some proposal pdf $g(\mathbf{x})$, the target probability is

$$R(u) = \int h_u(\mathbf{x})\frac{f(\mathbf{x})}{g(\mathbf{x})}g(\mathbf{x})d\mathbf{x} = \mathbb{E}_g\left[h_u(\mathbf{X})\frac{f(\mathbf{X})}{g(\mathbf{X})}\right], \tag{8}$$

where $\mathbb{E}_g$ denotes the expectation with respect to the new pdf g. Thus,

$$Z_u^{\text{IS}} = h_u(\mathbf{X})\frac{f(\mathbf{X})}{g(\mathbf{X})}, \quad \mathbf{X} \sim g, \tag{9}$$

is the unbiased estimator of $R(u)$.

The main problem arising here is how to choose the proposal distribution g in order to reduce the variance of the estimator Z_u^{IS}. It is quite straightforward to show (see for example [24]) that the optimal density g_* which provides the zero variance of the estimator has the following form:

$$g_*(\mathbf{x}) = \frac{h_u(\mathbf{x})f(\mathbf{x})}{R(u)}. \tag{10}$$

However, it is not implementable in practice because it requires knowledge of the target quantity $R(u)$. Nevertheless, sometimes it is possible to find a precise approximation of the optimal density g_*.

3.1.1 Cross-entropy method

The aim of this method is to find the proposal distribution g close to the desired zero-variance distribution g_* in the sense of the Kullback-Leibler divergence defined as [25, 26]

$$\begin{aligned}\mathcal{D}(g_*,g) &= \mathbb{E}_{g_*}\left[\log\frac{g_*(\mathbf{X})}{g(\mathbf{X})}\right] \\ &= \int g_*(\mathbf{x})\log g_*(\mathbf{x})d\mathbf{x} - \int g_*(\mathbf{x})\log g(\mathbf{x})d\mathbf{x}.\end{aligned}$$

When both proposal and nominal distributions are selected from some parametric set of distributions, this leads to the finite dimensional optimization problem. For this reason let's consider the parametric class of the multivariate normal distributions $f(\cdot;\boldsymbol{\theta})$ with parameters $\boldsymbol{\theta} = \{\mathbf{v}, \sigma\}$, i. e., with the mean vector $\mathbf{v} \in \mathbb{R}^L$ and covariance matrix $\mathbf{\Sigma}' = \sigma\mathbf{\Sigma}$, where $\mathbf{\Sigma} \in \mathbb{R}^{L\times L}$ is defined by (7) and $\sigma > 0$ is a scaling parameter.

The nominal pdf f defined by (6) belongs to this parametric class with $\boldsymbol{\theta}_0 = \{\mathbf{0}, 1\}$. Let's further choose the proposal density g from the same family with another parameter vector $\boldsymbol{\theta}$ further referred as a reference parameter. The cross-entropy (CE) method is based on finding an optimal reference parameter:

$$\begin{aligned}\boldsymbol{\theta}^* := \{\mathbf{v}^*, \sigma^*\} &= \operatorname*{argmin}_{\boldsymbol{\theta}} \mathcal{D}(g_*, f(\cdot;\boldsymbol{\theta})) \\ &= \operatorname*{argmax}_{\boldsymbol{\theta}} \mathbb{E}_{\boldsymbol{\eta}} \left[h_u(\mathbf{X}) W(\mathbf{X};\boldsymbol{\theta}_0,\boldsymbol{\eta}) \log f(\mathbf{X};\boldsymbol{\theta}) \right],\end{aligned}$$

where $\mathbb{E}_{\boldsymbol{\eta}}$ denotes the expectation with respect to the distribution $f(\cdot;\boldsymbol{\eta})$ and

$$W(\mathbf{X};\boldsymbol{\theta}_0,\boldsymbol{\eta}) = \frac{f(\mathbf{X};\boldsymbol{\theta}_0)}{f(\mathbf{X};\boldsymbol{\eta})}.$$

The given above stochastic optimization problem is replaced by its stochastic counterpart:

$$\boldsymbol{\theta}^* = \operatorname*{argmax}_{\boldsymbol{\theta}} \frac{1}{M} \sum_{k=1}^{M} h_u(\mathbf{X}^k) W(\mathbf{X}^k,\boldsymbol{\theta}_0,\boldsymbol{\eta}) \log f(\mathbf{X}^k;\boldsymbol{\theta}), \tag{11}$$

where $\mathbf{X}^1,..,\mathbf{X}^M$ is the sample from the distribution $f(\cdot;\boldsymbol{\eta})$. Setting the gradient of (11) with respect to $\mathbf{v}$ and σ to zero it is straightforward to obtain that

$$\mathbf{v}^* = \frac{\sum_{i=1}^{M} h_u(\mathbf{X}^i) W(\mathbf{X}^i;\boldsymbol{\theta}_0,\boldsymbol{\eta}) \mathbf{X}^i}{\sum_{i=1}^{M} h_u(\mathbf{X}^i) W(\mathbf{X}^i;\boldsymbol{\theta}_0,\boldsymbol{\eta})}, \tag{12}$$

$$\sigma^* = \frac{\sum_{i=1}^{M} h_u(\mathbf{X}^i) W(\mathbf{X}^i;\boldsymbol{\theta}_0,\boldsymbol{\eta}) (\mathbf{X}^i - \mathbf{v}^*)^T \boldsymbol{\Sigma}^{-1} (\mathbf{X}^i - \mathbf{v}^*)}{L \sum_{i=1}^{M} h_u(\mathbf{X}^i) W(\mathbf{X}^i;\boldsymbol{\theta}_0,\boldsymbol{\eta})}. \tag{13}$$

The main problem arising here is that most values of the $h_u(\mathbf{X}^i)$ are zero. In this case the so-called multi-level procedure [25, 26] can be applied. According to this approach the sequence $(u_t, \boldsymbol{\theta}_t)$ of both rarity and reference parameters is constructed, such that $\boldsymbol{\theta}_t$ is a solution of the problem (11) with the previous value of the reference parameter, i. e. $\boldsymbol{\eta} = \boldsymbol{\theta}_{t-1}$.

To be more precise, we start from the nominal vector of the parameters $\boldsymbol{\theta}_0$. Let further $(u_{t-1}, \boldsymbol{\theta}_{t-1})$ be the current values of the rarity and reference parameters respectively. The subsequent value of u_t is obtained by drawing a sample $\mathbf{X}^1,..,\mathbf{X}^M$ from the distribution $f(\cdot;\boldsymbol{\theta}_{t-1})$ as follows [27]:

$$u_t = T_D^{((1-\alpha)M)}, \tag{14}$$

where $T_D^{(j)}$ is the j-th order-statistics of the sequence $T_D(\mathbf{X}^1),...,T_D(\mathbf{X}^M)$; α is a free parameter chosen not very small (the typical value in practice is $\alpha = 0.05$).

Then, the next value of the reference parameter $\boldsymbol{\theta}_t$ is derived as a solution of the following CE program:

$$\boldsymbol{\theta}_t = \operatorname*{argmax}_{\boldsymbol{\theta}} \mathbb{E}_{\boldsymbol{\theta}_{t-1}} \left[h_{u_t}(\mathbf{X}) W(\mathbf{X};\boldsymbol{\theta}_0,\boldsymbol{\theta}_{t-1}) \log f(\mathbf{X};\boldsymbol{\theta}) \right].$$

Thus, the solution of the corresponding stochastic counterpart has the form (12)-(13) with $\boldsymbol{\eta} = \boldsymbol{\theta}_{t-1}$ and $u = u_t$.

The described above iterative algorithm terminates when $u_t \geq u$. After that, the target quantity is estimated as

$$\widehat{R}_u^{\mathrm{CE}} := \frac{1}{N} \sum_{i=1}^{N} h_u(\mathbf{X}^i) W(\mathbf{X}^i;\boldsymbol{\theta}_0,\boldsymbol{\theta}_S), \quad \mathbf{X}^1,...,\mathbf{X}^N \sim f(\cdot;\boldsymbol{\theta}_S), \tag{15}$$

where S denotes the last iteration number. The resulting CE procedure is summarized in the Algorithm 1 below.

Algorithm 1 The main CE algorithm

$\boldsymbol{\theta}_0 \leftarrow \{\mathbf{0}, 1\}$
$u_0 \leftarrow 0$
$t \leftarrow 0$
while $u_t < u$ **do**
  $t \leftarrow t+1$
  Generate a sample $\mathbf{X}^1, ..., \mathbf{X}^M \sim f(\cdot, \boldsymbol{\theta}_{t-1})$
  Compute u_t as (14)
  Compute $\boldsymbol{\theta}_t$ as a solution of CE program (12)-(13) with $\boldsymbol{\eta} = \boldsymbol{\theta}_{t-1}$ and $u = u_t$
end while
Compute the reliability according to (15)

Remark 1. Instead of estimating the mean vector $\mathbf{v}$ and the scaling factor σ of the covariance matrix one can try to estimate the full covariance matrix, namely choosing the parametrization as $\boldsymbol{\theta} = \{\mathbf{v}, \boldsymbol{\Sigma}\}$, where both mean vector $\mathbf{v}$ and covariance matrix $\boldsymbol{\Sigma}$ are estimated from the optimization problem (11). But the dimension of the corresponding optimization problem significantly increases, thus the number of samples M in (11) should be increased appropriately, otherwise, the overfitting problem can occur.

4. Simulation Results

In this section, we provide a simulation analysis of the accuracy of the proposed estimator.

All experiments were conducted for the case when the process X is the fractional Brownian motion (FBM) with the covariance function

$$\Gamma(t,s) := \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

Now we describe in brief the simulation procedure. First note that it is enough to simulate the FBM over the interval $[0, u]$. Sample paths of the FBM are drawn as realizations of the random vector:

$$(X(t_1), ...X(t_L)),$$

where $t_1, ..., t_L$ is a uniform partition of the interval $[0, u]$.

In all experiments performed below $N = 10000$ trajectories of the FBM with Hurst parameter $H = 0.7$ were generated.

The first numerical experiment deals with the case of a single change-point:

$$m(t) = \begin{cases} m_1, & t < \tau, \\ m_2, & t \geq \tau, \end{cases} \tag{16}$$

where

$$\tau = \min\{t : \ A(t) \geq D_1\},$$

and $D_1 < D$ is a given intermediate threshold.

The following values of the other parameters were used: $m_1 = 1, m_2 = 3$; $D_1 = 10$, $D = 20$. The number of samples M in (11) is 10^4. To verify the accuracy of the proposed estimators, we considered the dependence of the relative error on the rarity parameter u for both CE and standard MC estimators. The numerical results are presented in Table 1. The obtained results demonstrate that the CE estimator significantly outperforms the standard MC one. Note that the relative error is also estimated. To study the variability of the RE we performed 100 simulation runs of the described above experiment for the fixed value of the rarity parameter $u = 150$. We repeat this procedure for both $M = 10^4$ and $M = 10^5$ samples in the CE optimization problem (11) and calculate the empirical distribution of the RE for both cases. The obtained results presented

Table 1: *Performance of the estimators in case of the single change-point defined as* (16)*:* $m_1 = 1$, $m_2 = 3$.

u	$\widehat{R}^{\mathrm{MC}}$	$\widehat{R}^{\mathrm{CE}}$	$\mathrm{RE}(\widehat{R}^{\mathrm{MC}})$	$\mathrm{RE}(\widehat{R}^{\mathrm{CE}})$
40	0.0112	0.0096	0.0939	0.0239
50	0.0028	0.0035	0.1887	0.0280
60	0.0021	0.0014	0.2179	0.0416
70	9e-04	6.9e-04	0.3332	0.0336
80	7e-04	3.3e-04	0.3778	0.0385
90	3e-04	1.7e-04	0.5772	0.0488
100	–	8.7e-05	–	0.0581
110	–	4.9e-05	–	0.0675
120	–	2.6e-05	–	0.0999
130	–	1.4e-05	–	0.1269
140	–	8.3e-06	–	0.1541
150	–	5.9e-06	–	0.1861
160	–	1.9e-06	–	0.1827
170	–	1.4e-06	–	0.1979
180	–	8.8e-07	–	0.2371
190	–	1.6e-07	–	0.3514

in Fig. 3 indicate that the behavior of the CE estimator is much more robust in the sense of the variance of the RE when $M = 10^5$ since the larger value of M leads to the more precise approximation of the zero-variance distribution and consequently to the smaller RE which has the order 10^{-2} in case of $M = 10^5$.

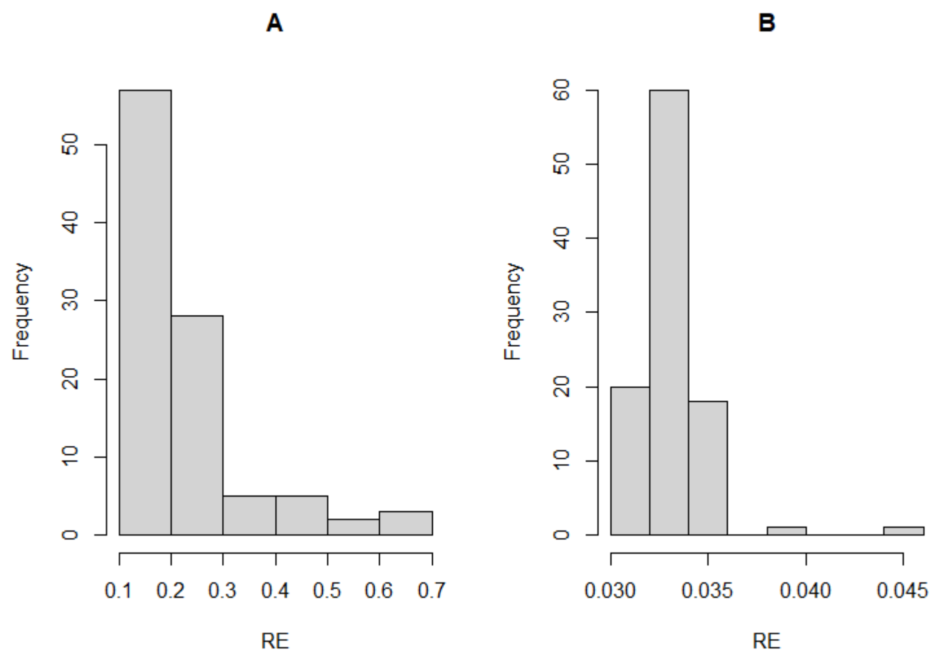

Figure 3: *Histogram of the RE in case of the degradation density defined as* (16)*: the number of samples used in CE optimization problem* (11) $M = 10^4$ (**A**)*,* $M = 10^5$ (**B**)*.*

For the second experiment, we consider the degradation intensity being dependent on the

Table 2: *Performance of the estimators for the degradation intensity defined as* (17)*:* $m_1 = 1$, $m_2 = 3$.

u	$\widehat{R}^{\text{MC}}$	$\widehat{R}^{\text{CE}}$	$\text{RE}(\widehat{R}^{\text{MC}})$	$\text{RE}(\widehat{R}^{\text{CE}})$
40	0.0092	0.0091	0.1037	0.0285
50	0.0055	0.0034	0.1344	0.0449
60	0.0021	0.0016	0.2179	0.0734
70	7e-04	7e-04	0.3778	0.0303
80	4e-04	3.5e-04	0.4999	0.0395
90	1e-04	1.7e-04	1	0.0412
100	–	8.7e-05	–	0.0606
110	–	5.7e-05	–	0.0881
120	–	2.4e-05	–	0.1102
130	–	1.4e-05	–	0.1466
140	–	9.8e-06	–	0.1387
150	–	3.7e-06	–	0.1247
160	–	3.2e-06	–	0.1535
170	–	1.6e-06	–	0.2080
180	–	6.8e-07	–	0.1847
190	–	4.9e-07	–	0.2229

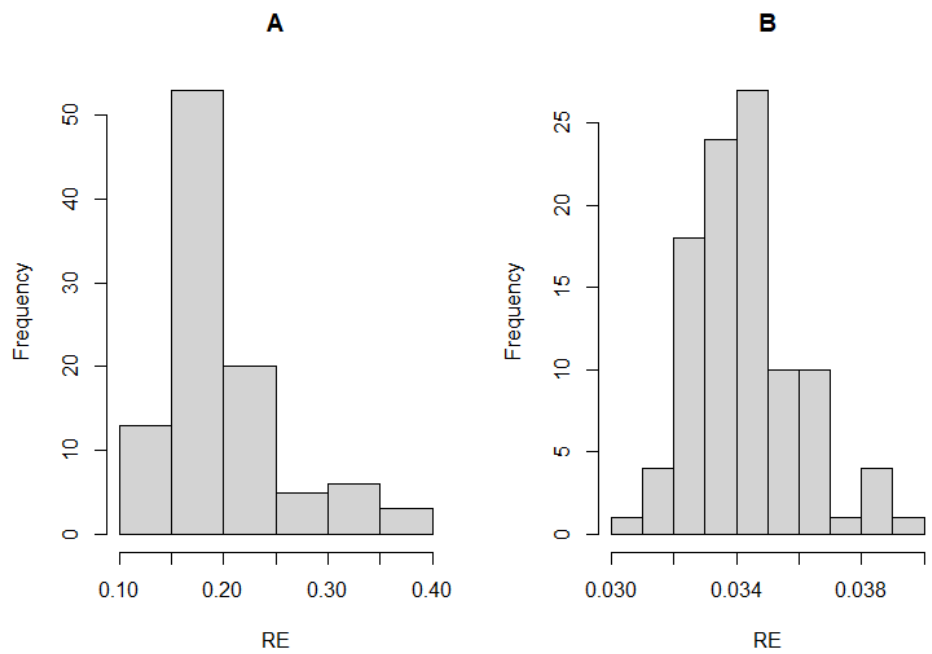

Figure 4: *Histogram of the RE in case of the degradation density defined as* (17)*: the number of samples used in CE optimization problem* (11) $M = 10^4$ (**A**), $M = 10^5$ (**B**).

degradation level at the previous time instant:

$$m(t) = \begin{cases} m_1, & A(t-1) < D_1, \\ m_2, & A(t-1) \geq D_1, \end{cases} \quad t \geq 1. \tag{17}$$

We conducted the same experiments with the same values of the parameters as a "sanity check". The results are summarized in Table 2 and Fig. 4. The results are quite similar to the ones obtained from the previous experiment since according to the definition (17) the switching time instant to the intensity m_2 will be close to the same value in the change-point model (16). Moreover, since $0 < m_1 < m_2$ the probability that the intensity will switch back to the level m_1 is rather small for FBM with $H = 0.7$.

5. Conclusion

In this paper, we have considered the Gaussian degradation model with the degradation intensity being dependent on the degradation history. Such a model has a very complicated dependence structure that makes it difficult to obtain the analytical expressions for the required performance measures. Thus, simulation remains the only available tool for analyzing such systems. To estimate the reliability of the considered degradation system the cross-entropy method has been applied which allows to calculate probabilities of rare events having a limited sample size. The numerical results indicate that the proposed estimator provides a significant reduction of the relative error in comparison with the standard Monte Carlo approach. The problem of accurate approximation of the zero-variance distribution seems to be an interesting topic of a further research.

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