RT&A, Special Issue, No 8 (85)
Volume 20, July 2025

REGENERATION AND APPROXIMATION OF A QUEUEING SYSTEM FED BY SUPERPOSED INPUT WITH WEIBULL COMPONENTS

Irina Peshkova, Michele Pagano, Evsey Morozov

•

Petrozavodsk State university, Russia
Institute of Applied Mathematical Research of the Karelian Research Centre of RAS,
Department of Information Engineering, University of Pisa, Italy
iaminova@petrsu.ru

Abstract

We study a single-server queueing system with a superposed input process formed by independent stationary renewal processes with Weibull interarrival time distributions. An approximating system with renewal input process based on Palm construction is considered. Moreover, the accuracy of the approximation in the terms of Kolmogorov distance is discussed. Finally, we demonstrate how to construct, in the initially non-regenerative queueing system, the artificial regenerations based on the exponential splitting technique.

Keywords: exponential splitting, regenerative estimation, Weibull distribution, superposition of renewal inputs

1. Introduction

In this research, we consider a single-server queueing system, denoted by $\sum_{i=1}^{m} GI_i/G/1$, in which the input process is a superposition of m independent stationary renewal processes (components). Such systems play an important role in modeling the modern communication systems, characterized by the superposition of independent heterogeneous traffic flows.

It is well-known that systems with the superposed input process are not (classically) regenerative unless the input components are Poisson processes. In particular, it is shown in [1] that such a superposed process obeys a weaker property, namely, the so-called *one-dependent regeneration*.

On the other hand, the presence of regenerative structure in the processes describing the dynamics of the system makes its analysis much more effective and accurate.

The study of systems with superposed input process has a long story, see for instance, the papers [2]- [3], in which various aspects of such systems are considered, including the covariance between interarrival times in the superposed process [2] and conditions for its convergence to a Poisson process [4, 5]. Moreover, a considerable attention has been paid to the analysis of the approximation of such a system by a classic $GI/G/1$ queueing system with renewal input, which, in turn, is constructed as a Palm stationary process composed from the (properly weighted) distributions of the interarrival times in the component processes [6]. In a companion paper [7], we studied such a system provided the interarrival times in the component processes have either exponential or Pareto distributions. The presence of the heavy-tailed distributions permits to apply the so-called exponential splitting [8] to construct classical regenerations in a modified system, which is equivalent, in the terms of marginal (one-dimensional) distributions, to the original one. This approach uses a possibility to split the corresponding 'heavy-tailed' density in such a way that it is represented as a two-component mixture containing (weighted) exponential density.

The main contribution of the present research is that we now consider, in the superposed input process, independent (stationary) heavy-tailed Weibull components, instead of the Pareto distribution used in the related paper [7]. We apply the exponential splitting to construct classical regenerations and, using regenerative method, obtain then the confidence interval for the mean stationary workload (unfinished work) in the system. One more contribution of this work is that, for such a system, we construct an approximating system based on the Palm theory and investigate the accuracy of this approximation using various metrics. Numerical results illustrating theoretical findings are included as well. In summary, in this paper we realize, in the main features, the research program which has been developed in our previous paper [7] for the superposed process containing Weibull distributed components.

The paper is organized as follows. In Section 2, we describe the basic model and an approximating renewal process, based on the Palm approach, with the same (one-dimensional) interevent distribution as in the original superposed process. Then we focus on the input process containing two independent renewal processes with Weibull interevent distributions. As we show, this approximating distribution is a two-component mixture, again containing Weibull distributions. Section 3 deals with the splitting procedure of the heavy-tailed Weibullian density, which then is used to construct regenerations of both the (superposed) input process and the processes describing the dynamics of the whole system. Section 4 contains simulation results related to the accuracy (in terms of Kolmogorov distance) of the approximation of the original system by a $GI/G/1$ queueing system (Section 4.1) and to the efficiency of the exponential splitting (Section 4.2), measured in the term of the frequency of the regenerations in a limited simulation time.

2. Basic model

We remind that the basic system is fed by a superposition of m independent renewal processes. We assume that the ith input component is a stationary renewal process, which is defined by the independent identically distributed (iid) interarrival times between customers (class-i ones) $\{\tau_k^{(i)}, k \geq 1\}$ with distribution function A_i and mean $\mathsf{E}\tau^{(i)} = 1/\lambda_i < \infty$, ($\tau^{(i)}$ is the generic interarrival time) and the time up to the 1st arrival has the following *integrated-tail distribution*

$$\lambda_i \int_0^x (1 - A_i(u))du, \;\; i = 1, \ldots, m.$$

It is well-known that such a choice guarantees stationarity of each component (renewal) process and as a result the stationarity of the superposed point process composed from the (ordered) points of the components [9] (Sect.3, Ch. 5.) The stationary interarrival time in the superposed (stationary) process follows the Palm distribution [3, 10]:

$$A(x) = 1 - \sum_{i=1}^{m} \frac{\lambda_i}{\lambda}(1 - A_i(x)) \prod_{j, j \neq i} \lambda_j \int_x^{\infty} (1 - A_j(x))dx, \tag{1}$$

where $\lambda = \lambda_1 + \cdots + \lambda_m$. We note that the (tail) distribution $1 - A(x)$ is a m-component mixture (with the weights $p_i := \lambda_i/\lambda$) of the original distributions describing interarrival times in the component renewal processes. Also denote by $\{S_k^{(i)}, k \geq 1\}$ the iid service times of the class-i customers (from the ith component process) with distribution function B_i and finite mean service rate $\mu_i := 1/\mathsf{E}S^{(i)}$, $i = 1, \ldots, m$. Then the service time distribution (of an arbitrary customer) in the system is also represented as the mixture

$$B(x) := \sum_{i=1}^{m} p_i B_i(x), \;\; x \geq 0. \tag{2}$$

The stability criterion of this system is well-known [6, 11]:

$$\rho = \sum_{i=1}^{m} \lambda_i \mathsf{E}S^{(i)} < 1, \tag{3}$$

which also is the stability criterion of the original $\sum_{i=1}^{m} GI/G/1$ system fed by a stationary input process [10]. Note that in the latter case, in the absence of regenerative structure, the proof of stability is based on construction proposed in [12]. We remark that, as it is easy to check, condition (3) can also be written as $\lambda ES < 1$, where S has distribution $B(x)$ from (2).

In what follows we will consider, as the main example, a superposed input composed by two independent stationary components with iid interarrival times $\{\tau_k^{(i)}, k \geq 1\}$ having Weibull distribution (denoted further by $We(\alpha_i, \beta_i)$) with parameters α_i, β_i:

$$A_i(x) = 1 - e^{-(x/\alpha_i)^{\beta_i}}, \;\; x \geq 0, \; \alpha_i > 0, \; \beta_i > 0, \;\; i = 1, 2. \tag{4}$$

Remark. The Weibull distribution was introduced and studied by several prominent researchers in the 1930s, besides W. Weibull, these include R. Fisher, L. Tippett, and L. von Mises. B. Gnedenko later has established the conditions under which a suitably normalized sequence of extreme values converges to one of three limiting distributions, including Weibull distribution, [13]. Because of this prominent research, the generalized two-parameter distribution (4) sometimes also referred to as the Weibull-Gnedenko distribution, see for instance, [14].

Substituting distributions (4) in formula (1), we obtain the distribution of the stationary *renewal input process* in the form:

$$A(x) = 1 - \frac{p}{\Gamma(1/\beta_2)} e^{-(x/\alpha_1)^{\beta_1}} \Gamma(1/\beta_2, (x/\alpha_2)^{\beta_2}) - \frac{1-p}{\Gamma(1/\beta_1)} e^{-(x/\alpha_2)^{\beta_2}} \Gamma(1/\beta_1, (x/\alpha_1)^{\beta_1}),$$

where

$$\Gamma(\xi, x) = \int_x^{\infty} e^{-t} t^{\xi - 1} dt$$

is the upper incomplete Gamma function,

$$p = \frac{\beta_1 \alpha_2 \Gamma(1/\beta_2)}{\beta_1 \alpha_2 \Gamma(1/\beta_2) + \beta_2 \alpha_1 \Gamma(1/\beta_1)}, \tag{5}$$

is the 'mixing proportion' and $\Gamma(\xi)$ is the Gamma function. Finally, applying the relation

$$\Gamma(\xi, x) = \Gamma(\xi) - \gamma(\xi, x),$$

where $\gamma(\xi, x)$ is the lower incomplete gamma function, we obtain the following (tail) Palm distribution of the renewal intervals in the $GI/G/1$ approximating system:

$$\begin{aligned} \overline{A}(x) \;\; &:= \;\; 1 - A(x) = p e^{-(x/\alpha_1)^{\beta_1}} \left(1 - \frac{1}{\Gamma(1/\beta_2)} \gamma(1/\beta_2, (x/\alpha_2)^{\beta_2})\right) + \\ &+ \;\; (1-p) e^{-(x/\alpha_2)^{\beta_2}} \left(1 - \frac{1}{\Gamma(1/\beta_1)} \gamma(1/\beta_1, (x/\alpha_1)^{\beta_1})\right). \end{aligned} \tag{6}$$

Notice that the tail distribution (6) is the mixture,

$$\overline{A}(x) = p\overline{F}_1(x) + (1-p)\overline{F}_2(x), \tag{7}$$

containing tail distributions

$$\begin{aligned} \overline{F}_1(x) &= e^{-(x/\alpha_1)^{\beta_1}} \left(1 - \frac{1}{\Gamma(1/\beta_2)} \gamma(1/\beta_2, (x/\alpha_2)^{\beta_2})\right), \\ \overline{F}_2(x) &= e^{(-x/\alpha_2)^{\beta_2}} \left(1 - \frac{1}{\Gamma(1/\beta_1)} \gamma(1/\beta_1, (x/\alpha_1)^{\beta_1})\right), \end{aligned} \tag{8}$$

with the mixing proportion p defined by (5).

Denote by τ the (generic) interarrival time with distribution (6). Let distribution F_1 define a random variable (r.v.) Y, and distribution F_2 define a r.v. Z. Then τ can be expressed by the two-component mixture as

$$\tau = IY + (1-I)Z, \tag{9}$$

where I is the indicator function with $\mathsf{P}(I=1) = p$ and

$$\begin{aligned} Y &= \min(Y_1, Y_2) \text{ with } Y_1 \sim We(\alpha_1, \beta_1),\ \ Y_2 \sim SGGa(\alpha_2, 1/\beta_2, \beta_2), \\ Z &= \min(Z_1, Z_2) \text{ with } \ Z_1 \sim We(\alpha_2, \beta_2),\ \ Z_2 \sim SGGa(\alpha_1, 1/\beta_1, \beta_1), \end{aligned}$$

where the symbol $\sim$ connects a r.v. and its distribution, and $SGGa(\alpha, a, c)$ denotes Stacy's generalized Gamma distribution

$$F(x) = \frac{1}{\Gamma(a)}\gamma(a, (x/\alpha)^c), \quad x \geq 0, \tag{10}$$

(where $a = 1/\beta_i, c = \beta_i, i = 1,2$), with density function

$$f(x) = \frac{c}{\alpha\Gamma(a)}(x/\alpha)^{ca-1}e^{-(x/\alpha)^c}, \quad x \geq 0,$$

and moments

$$EX = \frac{\alpha\Gamma(a+1/c)}{\Gamma(a)}, \ \ EX^2 = \frac{\alpha^2\Gamma(a+2/c)}{\Gamma(a)}.$$

In this setting, the stability condition (3) of approximating system $GI/G/1$ becomes

$$\rho = \frac{\beta_1}{\alpha_1\Gamma(1/\beta_1)}\mathsf{E}S^{(1)} + \frac{\beta_2}{\alpha_2\Gamma(1/\beta_2)}\mathsf{E}S^{(2)} < 1.$$

Below we illustrate the analysis by numerical results based on Monte-Carlo simulation and regenerative approach.

3. Construction of classical regeneration by exponential splitting

In this section we discuss the application of exponential splitting to regenerative simulation of queues with superposed heavy-tailed Weibull inputs. This approach is based on the memoryless property of exponential distribution and the synchronization of a regeneration point of the input process and an empty state of the system. When the component processes have Pareto interevent distributions, this approach has been developed in a recent paper [7].

As far as the authors know, the idea of exponential splitting has been firstly described in [8]. It involves replacing the original r.v. T by a stochastically equivalent r.v. T' (defined on an enlarged probability space). More exactly, an absolutely continuous positive r.v. T with the density g is called *exponentially split* if there exist constants $\eta > 0$ and $\delta \in (0,1)$ such that

$$g(x) \geq \delta\eta e^{-\eta x}, \quad x \geq 0. \tag{11}$$

Let us define a new r.v. T' as follows:

$$T' = I_T T_0 + (1-I_T)T_1, \tag{12}$$

where the r.v. T_0 has density $g_0(x) = \eta e^{-\eta x}$, the r.v. T_1 has density

$$g_1(x) = \frac{g(x) - \delta g_0(x)}{1-\delta}, \tag{13}$$

and I_T is the Bernoulli r.v. (called *splitting indicator*) such that $\mathsf{P}(I_T = 1) = \delta$. If the event $\{I_T = 1\}$ happens, we say that the exponential phase takes place. For Weibull distribution (4), the inequality (11) transforms into

$$\frac{\beta}{\alpha}(x/\alpha)^{\beta-1}e^{-(x/\alpha)^\beta} \geq \delta\eta e^{-\eta x}, \tag{14}$$

which indeed holds under the following conditions connecting the parameters:

$$0 < \delta \le \beta/\alpha, \quad \eta = \alpha^{-\beta},\ 0.5 \le \beta < 1,\ \alpha \ge 1. \tag{15}$$

In our example the r.v. T_1 with the density (13) has distribution function

$$G_1(x) = 1 - \frac{1}{1-\delta} e^{-(x/\alpha)^\beta} + \frac{\delta}{1-\delta} e^{-\eta x}, \ \ x \ge 0.$$

From the simulation viewpoint, splitting means that, instead of generating an r.v. T with Weibull distribution, a triple (T_0, T_1, I_T) is generated, where T_0 is exponential with parameter η, T_1 has distribution function G_1 and I_T is the splitting indicator. Moreover, the inequalities (15) ensure that the basic inequality (11) holds. We note that (11) is a particular case of the so-called *minorization condition*, which plays an important role in the theory of general Markov chains, see for instance, [15].

To construct the regeneration points for the waiting time process $W = \{W_n, n \ge 1\}$ in the $\sum_{i=1}^m GI/G/1$ system, we denote by $\{t_k^{(i)}\}$ the arrival instances of the i-th input and by $\{t_k\}$ the arrival points in the superposed input process.

Now we select and fix an arbitrary component input process, denoted further by i_0, with the arrival instances $\{t_k^{(i_0)} \equiv t_k^{(0)}, k \ge 1\}$. Also define index $n(k)$ as $t_k = t_{n(k)}^{(0)}$. In other words, the k-th arrival in the superposed input (at instant t_k) is indeed the $n(k)$-th class-i_0 arrival. Let indicator function $I_j(t) = 1$ if, at instant t, the j-th component process is in the exponential phase (see decomposition (12)), and $I_j(t) = 0$, otherwise. Now define the following events:

$$\mathcal{E}_k^{(0)} = \{I_j(t_k^{(0)}) = 1,\ j = 1, \ldots, m;\ j \ne i_0)\},\ \ k \ge 1. \tag{16}$$

In other words, the event $\mathcal{E}_k^{(0)}$ means that, at the arrival instant $t_k^{(0)}$ of a class-i_0 customer, the interarrival times of all inputs $j \ne i_0$ have exponential phase. It is easy to check that, on the event $\mathcal{E}_k^{(0)}$, the superposed input process classically regenerates. To illustrate it, we return to the basis model (see Section 2) in which the interarrival time in the i-th renewal input has distribution A_i, and let its exponential phase have parameter λ_i. Then, on the event $\mathcal{E}_k^{(0)}$, the remaining time up to the next event in the superposed process at the arrival epoch $t_k^{(0)}$ of a class-i_0 customer, denoted by $\tau(t_k^{(0)})$, has (tail) distribution

$$\mathsf{P}(\tau(t_k^{(0)}) > x) = (1 - A_{i_0}(x)) \exp\{-x \sum_{j \ne i_0} \lambda_j\},\ \ x \ge 0,$$

which is independent of instant $t_k^{(0)}$. Thus, the regeneration instances $\{\gamma_n\}$ of the superposed input process can be defined as follows:

$$\gamma_0 = 0,\ \ \gamma_{k+1} = \min\{i : 1(t_{n(i)}^{(0)} > \gamma_k) \cdot 1(\mathcal{E}_{n(i)}^{(0)}) = 1\},\ k \ge 0. \tag{17}$$

To construct regenerations of the entire model (that is, the basic processes describing the dynamics of the system) we need one more step. Take the waiting time process $W = \{W_n, n \ge 1\}$ as a basic process. Then it *classically* regenerates at arrival instant of a class-i_0 customer, if i) it is a regeneration instant of the superposed input process and ii) the system is idle at this instant. Formally, these instants can be defined recursively as

$$\beta_0 = 0,\ \beta_{k+1} = \min\{\gamma_i > \beta_k : W_{\gamma_i} = 0\},\ k \ge 0. \tag{18}$$

Then the random distances $\widehat{\Delta}_k := \beta_{k+1} - \beta_k$, $k \ge 1$, are the iid regeneration cycle lengths.

In what follows we simulate and estimate by the regenerative method [16] the stationary performance of the waiting time process W when the superposed input process is composed by two components with heavy-tailed Weibull distributions.

4. Simulation results

The aim of this Section is to investigate through discrete event simulation the two main theoretical contributions of this paper. At first, Section 4.1 deals with the renewal approximation of the original system fed by two independent Weibull flows with different values of β_i (including the case $\beta_i = 1$ that corresponds to Poisson arrivals). Then, in Section 4.2 we focused on the heavy-tailed case (i.e., $\beta_i < 1$) and evaluated the efficiency of the exponential splitting in terms of the frequency of the regenerations for different values of the splitting parameters.

4.1. Approximation by $GI/G/1$

We consider a queueing systems fed by two independent input processes with Weibull interarrival times. In all the simulations described in this subsection 10^{10} arrivals are generated and common (class-independent) exponential service times with rate μ are assumed.

We investigate the accuracy of the approximation through the Kolmogorov distance $d(W, W_A)$ between the empirical distributions of the stationary waiting time process W in the basic system $\sum_{i=1}^{2} We(\alpha_i, \beta_i)/M/1$ and the waiting time process W_A in the approximating system $GI/M/1$ for different values of traffic intensity ρ.

The first set of simulations is carried out with the following values of the parameters: $\alpha_1 = 1$, $\alpha_2 = 4$, $\beta_1 = \beta_2 = 1$, which corresponds to the superposition of two Poisson processes. It is easy to check that in this case the approximating system is an $M/M/1$ system with Poisson input with parameter $1/\alpha_1 + 1/\alpha_2$. We can compare simulation results of the actual waiting time trajectories for the original system W and the waiting time W_A in the corresponding $M/M/1$ queueing system. The absolute error does not exceed 8×10^{-5} for all values of traffic intensity $0.1 < \rho < 0.9$.

The next two sets of simulations investigates the goodness of the approximation in case of light-tailed ($\beta_i > 1$)and heavy-tailed ($\beta_i < 1$) Weibull distributions, respectively.

The comparison between W and W_A (see Tables 1 and 2, where the simulation settings are also reported) includes not only the Kolmogorov distance, but also the values of the average workload process and its variance in the two cases. It is worth noticing that in the first case (see Table 1), the Kolmogorov distance $d(W, W_A)$ does not exceed 5% only for values $\rho < 0.5$, while in presence of heavy-tailed Weibull distributions (see Table 2) the Kolmogorov distance $d(W, W_T)$ is very small for any value of traffic intensity ρ.

$\beta_1 = 5, \beta_2 = 10,\ \alpha_1 = \alpha_2 = 1$					
ρ	$d(W, W_A)$	$Mean(W)$	$Mean(W_A)$	$Var(W)$	$Var(W_A)$
0.1	0.00277	0.00234	0.00260	0.00021	0.00025
0.2	0.01181	0.00948	0.01189	0.00169	0.00237
0.3	0.02499	0.02295	0.03152	0.006107	0.00983
0.4	0.03849	0.04763	0.06805	0.01745	0.03007
0.5	0.05085	0.09289	0.13311	0.04568	0.07991
0.6	0.06189	0.17661	0.24946	0.11651	0.20209
0.7	0.07168	0.33954	0.46978	0.30896	0.52809
0.8	0.08027	0.70258	0.95107	0.95681	1.61544
0.9	0.08808	1.87013	2.47942	4.89730	8.23029

Table 1: *Simulation results* $\beta_1 = 5, \beta_2 = 10,\ \alpha_1 = \alpha_2 = 1.$

$\beta_1 = 0.1, \beta_2 = 0,2, \alpha_1 = \alpha_2 = 1$					
ρ	$d(W, W_A)$	$Mean(W)$	$Mean(W_A)$	$Var(W)$	$Var(W_A)$
0.1	1.909×10^{-5}	95.0969	95.0917	11325.2	11322.9
0.2	8.228×10^{-5}	355.945	355.870	143765	143709
0.3	9.561×10^{-5}	874.066	873.498	827005	825552
0.4	2.952×10^{-5}	1827.71	1827.88	$3.52\times10^{+6}$	$3.52\times10^{+6}$
0.5	11.72×10^{-5}	3572.09	3571.48	$1.32\times10^{+7}$	$1.32\times10^{+7}$
0.6	20.18×10^{-5}	6869.13	6865.69	$4.82\times10^{+7}$	$4.81\times10^{+7}$
0.7	13.88×10^{-5}	13652.6	13652.4	$1.89\times10^{+8}$	$1.89\times10^{+8}$
0.8	69.37×10^{-5}	30214.2	30144.3	$9.19\times10^{+8}$	$9.13\times10^{+8}$
0.9	14.63×10^{-4}	90455.3	90122.8	$8.19\times10^{+9}$	$8.11\times10^{+9}$

Table 2: *Simulation results* $\beta_1 = 0.1, \beta_2 = 0,2,\ \alpha_1 = \alpha_2 = 1.$

4.2. Artificial regeneration by exponential splitting

In this subsection we use several sets of simulation (with 10^9 arrivals) to evaluate the influence of the selected process i_0, splitting parameter δ, exponential parameter η and the traffic intensity ρ, see minorization inequality (11). We focus on the heavy-tailed case, namely, with $\alpha_1 = 1.5$, $\beta_1 = 0.5$, $\alpha_2 = 2$ and $\beta_2 = 2/3$.

As regenerative simulation allows us to calculate confidence intervals, in the following tables we report (for different values of ρ) not only the mean value of the waiting time process W, Mean(W), but also the half-width of the 99% confidence interval, 99%CI, and the number of regeneration cycles, RegW, of the queueing system. It is worth noticing that the number of regenerations in the superposed input (i.e., the number of events (16)), RegIn, does not depend on ρ, while it is proportional to δ.

Exponential splitting, $i_0 = 2$

ρ	$\delta = 0.33$			$\delta = 0.1$		
	RegW	Mean(W)	99%CI	RegW	Mean(W)	99%CI
0.1	5.2×10^7	0.048	1.8×10^{-5}	1.5×10^7	0.048	1.8×10^{-5}
0.2	3.8×10^7	0.184	7.1×10^{-5}	1.1×10^7	0.184	7.1×10^{-5}
0.3	2.7×10^7	0.439	18.5×10^{-5}	8.3×10^6	0.439	18.5×10^{-5}
0.4	1.9×10^7	0.875	41.7×10^{-5}	5.8×10^6	0.875	41.6×10^{-5}
0.5	1.3×10^7	1.604	89.3×10^{-5}	4.0×10^6	1.604	89.2×10^{-5}
0.6	8.9×10^6	2.852	19.4×10^{-4}	2.7×10^6	2.854	19.4×10^{-4}
0.7	5.6×10^6	5.162	45.6×10^{-4}	1.7×10^6	5.162	4.6×10^{-3}
0.8	3.2×10^6	10.139	13.1×10^{-3}	9.6×10^5	10.142	13.2×10^{-3}
0.9	1.4×10^7	25.905	65.6×10^{-3}	4.1×10^5	25.885	64.7×10^{-3}
0.92	1.0×10^6	33.977	10.7×10^{-2}	3.1×10^5	33.927	10.6×10^{-2}
0.95	6.2×10^5	58.202	28.9×10^{-2}	1.8×10^5	58.190	25.3×10^{-2}

Table 3: *Simulation results for* $\eta = 0.8164, \beta_1 = 0.5, \beta_2 = 0.67, \alpha_1 = 1.5, \alpha_2 = 2.$

At first we select the second component of the superposed process as i_0-process, so the

RT&A, Special Issue, No 8 (85)
Volume 20, July 2025

Exponential splitting, $i_0 = 1$

ρ	$\delta = 0.33$			$\delta = 0.1$		
	RegW	Mean(W)	99%CI	RegW	Mean(W)	99%CI
0.1	6.2×10^7	0.048	1.8×10^{-5}	1.85×10^7	0.048	1.8×10^{-5}
0.2	4.6×10^7	0.184	7.1×10^{-5}	1.38×10^7	0.184	7.1×10^{-5}
0.3	3.4×10^7	0.439	18.5×10^{-5}	1.03×10^7	0.439	18.5×10^{-5}
0.4	2.5×10^7	0.875	41.7×10^{-5}	7.63×10^6	0.875	41.7×10^{-5}
0.5	1.8×10^7	1.604	89.2×10^{-5}	5.50×10^6	1.604	89.3×10^{-5}
0.6	1.3×10^7	2.853	19.3×10^{-4}	3.818×10^6	2.855	19.4×10^{-4}
0.7	8.3×10^6	5.161	4.5×10^{-3}	2.49×10^6	5.162	45.6×10^{-4}
0.8	4.8×10^6	10.147	1.3×10^{-2}	1.45×10^6	10.148	13.1×10^{-3}
0.9	2.1×10^6	25.855	6.5×10^{-2}	6.33×10^5	25.904	65.5×10^{-2}
0.92	1.6×10^6	33.956	0.107	4.93×10^5	33.903	10.7×10^{-2}
0.95	9.8×10^5	58.278	0.291	2.96×10^5	58.275	29.4×10^{-2}

Table 4: *Simulation results for* $\eta = 0.63,\ \beta_1 = 0.5, \beta_2 = 0.67, \alpha_1 = 1.5, \alpha_2 = 2.$

exponential splitting is applied to the first flow. From the inequalities (15) it is easy to see that, choosing $\eta = 1/\sqrt{1.5}$, the exponential splitting is possible for $0 < \delta \le 1/3$. For sake of brevity, the simulation results are reported only for two values of δ.

As highlighted in Table 3, when the traffic intensity ρ increases, RegW decreases, while Mean(W) and 99%CI increase. The value of δ does not significantly affect the reported parameters, apart from RegW. In more detail, for $\delta = 0.33$ RegIn $\approx 7.2 \times 10^7$, while for $\delta = 0.1$ the value of RegIn $\approx 2.16 \times 10^7$ is around ≈ 3.3 times less. Similar consideration may be applied to the values of RegW.

Similar considerations can be drawn when the exponential splitting is applied to the second component (see Table 4). In this case, according to the inequalities (15), we take $\eta = 2^{-2/3}$ and the exponential splitting is still possible for $0 < \delta \le 1/3$. Note that in this case the input process regenerates more frequently, but the ratio between the number of regenerations and the value of δ remains almost the same (namely, RegIn $\approx 9.3 \times 10^7$ and RegIn $\approx 2.8 \times 10^7$ for $\delta = .33$ and $\delta = 0.1$, respectively).

5. Conclusion

In this paper we analyzed a queueing system fed by the superposition of independent stationary renewal processes with Weibull interarrival time distribution, focusing on two relevant issues: the renewal approximation of the input process and the regenerative simulation of the original queueing system.

In more detail, the accuracy of the renewal approximation was evaluated in terms of the Kolmogorov distance for the corresponding distributions of the stationary workload (the remaining work to be processed). Simulation experiments pointed out that the approximation works perfectly for any value of traffic intensity ρ in the case of heavy-tailed Weibull components and only for $\rho < 0.6$ in the light-tail case. This unexpected result needs further investigation and explanation.

Moreover, the exponential splitting was applied to construct artificial regenerations and estimate the mean stationary workload (with the confidence interval) in the basic (non-regenerative) system. Sufficient conditions for the applicability of the exponential splitting have been derived in the case of heavy-tailed Weibull interarrival time distributions and the theoretical results have been verified by simulation.

We demonstrated the efficiency of the approach based on the construction of regenerations using exponential splitting technique. It seems to be highly effective for the queueing systems containing heavy-tailed distributions. In this regard the authors hope that this research (as well as the related previous paper [7]) creates a methodological basis for reliable regeneration-based estimation of a broad class of the systems described by heavy-tailed distributions.

Acknowledgments

This work was partially supported by the Moscow Center for Fundamental and Applied Mathematics (recipient E. Morozov).

References

[1] Karl Sigman. "One-Dependent Regenerative Processes and Queues in Continuous Time". In: *Mathematics of Operations Research* 15.1 (1990), pp. 175–189. URL: http://www.jstor.org/stable/3689937 (visited on 06/06/2025).

[2] S. L. Albin. "On Poisson Approximations for Superposition Arrival Processes in Queues". In: *Management Science* 28.2 (Feb. 1982), pp. 126–137. DOI: 10.1287/mnsc.28.2.126.

[3] Ward Whitt. "Approximating a Point Process by a Renewal Process, I: Two Basic Methods". In: *Operations Research* 30.1 (Feb. 1982), pp. 125–147. DOI: 10.1287/opre.30.1.125.

[4] A.Ia. Khinchin. *Mathematical Methods in the Theory of Queueing*. Griffin's Statistical Monographs. London, Griffin, 1960. ISBN: 978-0-02-847880-7.

[5] G. F. Newell. "Approximations for Superposition Arrival Processes in Queues." In: *Management Science* 30.5 (1984), pp. 623–632. (Visited on 06/06/2025).

[6] Søren Asmussen, Hanspeter Schmidli, and Volker Schmidt. "Tail probabilities for non-standard risk and queueing processes with subexponential jumps". In: *Advances in Applied Probability* 31.2 (June 1999), pp. 422–447. DOI: 10.1239/aap/1029955142.

[7] Irina Peshkova, Evsey Morozov, and Michele Pagano. "Regenerative Analysis and Approximation of Queueing Systems with Superposed Input Processes". In: *Mathematics* 12.14 (July 2024), p. 2202. DOI: 10.3390/math12142202.

[8] Alexander Andronov. "Artificial regeneration points for stochastic simulation of complex systems". In: *Simulation Technology: Science and Art. 10th European Simulation Symposium ESS'98*. SCS, Delft, The Netherlands, 1998, pp. 34–40.

[9] Søren Asmussen. *Applied probability and queues*. English. New York: Springer, 2003. ISBN: 978-0-387-00211-8.

[10] François Baccelli and Pierre Brémaud. *Elements of Queueing Theory*. Springer Berlin Heidelberg, 2003. ISBN: 9783662116579. DOI: 10.1007/978-3-662-11657-9.

[11] Evsey Morozov and Bart Steyaert. *Stability Analysis of Regenerative Queueing Models: Mathematical Methods and Applications*. Springer International Publishing, 2021. ISBN: 9783030824389. DOI: 10.1007/978-3-030-82438-9.

[12] R. M. Loynes. "The stability of a queue with non-independent inter-arrival and service times". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 58.3 (July 1962), pp. 497–520. DOI: 10.1017/s0305004100036781.

[13] B. Gnedenko. "Sur La Distribution Limite Du Terme Maximum D'Une Série Aléatoire". In: *Annals of Mathematics* 44.3 (1943), pp. 423–453. URL: http://www.jstor.org/stable/1968974 (visited on 06/06/2025).

[14] Frank Beichelt and Peter Franken. *Zuverlässigkeit und Instandhaltung: mathematische Methoden Instandhaltung, Verfügbarkeit, Zuverlässigkeit*. Vol. 1. Veb Verlag Technik, 1983.

[15] Sean P. Meyn and Richard L. Tweedie. *Markov Chains and Stochastic Stability*. Springer London, 1993. ISBN: 9781447132677. DOI: 10.1007/978-1-4471-3267-7.

[16] Søren Asmussen and Peter W. Glynn. *Stochastic simulation: algorithms and analysis*. en. Stochastic modelling and applied probability 57. OCLC: ocn123113652. New York: Springer, 2007. ISBN: 978-0-387-69033-9.